Politecnico di Torino

Bridging course in mathematics

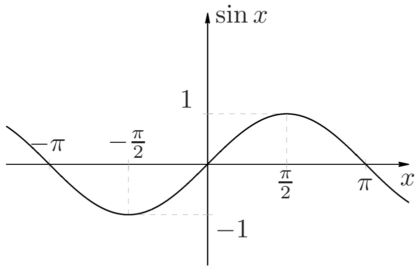
Sheet 5

Trigonometry

1 Trigonometric maps

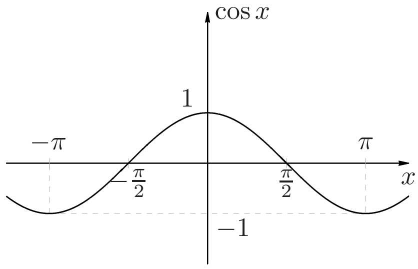
1.1 The sine function

The map $y = \sin x$ has domain $\mathbb{R}$ and range $[-1, 1]$. It is periodic of period 2π. Relatively to the interval $[0, 2\pi]$ we have $\sin(0) = \sin(\pi) = 0$; $\sin\left(\frac{\pi}{2}\right) = 1$; $\sin\left(\frac{3\pi}{2}\right) = -1$. The function $\sin x$ is positive on $(0, \pi)$ and negative on $(\pi, 2\pi)$; it is increasing on $\left[0, \frac{\pi}{2}\right]$ and on $\left[\frac{3}{2}\pi, 2\pi\right]$, while decreasing on $\left[\frac{\pi}{2}, \frac{3}{2}\pi\right]$. The function is odd, in other words $\sin(-x) = -\sin(x)$.



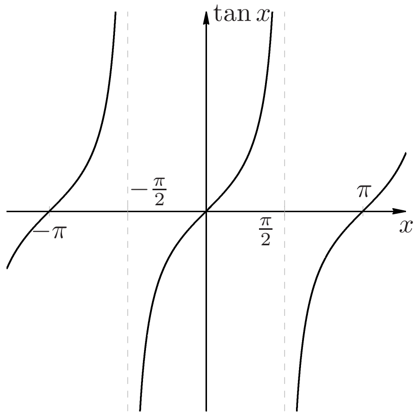
1.2 The cosine function

The map $y = \cos x$ has domain $\mathbb{R}$ and range $[-1, 1]$. It is periodic of period 2π. Relatively to $[0, 2\pi]$ we have $\cos(0) = 1$, $\cos\left(\frac{\pi}{2}\right) = \cos\left(\frac{3}{2}\pi\right) = 0$, $\cos(\pi) = -1$. It is positive on $\left(0, \frac{\pi}{2}\right) \cup \left(\frac{3}{2}\pi, 2\pi\right)$, negative on $\left(\frac{\pi}{2}, \frac{3}{2}\pi\right)$, increasing on $[\pi, 2\pi]$ and decreasing on $[0, \pi]$. This map is even, since $\cos(-x) = \cos(x)$.



1.3 The tangent function

The tangent is defined by $y = \tan x = \frac{\sin x}{\cos x}$. Another notation is $y = \operatorname{tg} x$. The domain is $\mathbb{R} \setminus \left\{x : \ x = \frac{\pi}{2} + k\pi, \ k \in \mathbb{Z}\right\}$, while the range is the entire real axis $\mathbb{R}$. It is periodic of period π. Relatively to $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ the map is positive on $\left(0, \frac{\pi}{2}\right)$, negative on $\left(-\frac{\pi}{2}, 0\right)$, it vanishes at $x = 0$ and is increasing. It is an odd map, as $\tan(-x) = -\tan(x)$.



1.4 Other trigonometric functions

The reciprocal functions of sine, cosine and tangent are respectively called *cosecant*, *secant* and *cotangent*:

- Cosecant: $\operatorname{cosec} x = \frac{1}{\sin x}$ for $x \neq k\pi, k \in \mathbb{Z}$;
- Secant: $\sec x = \frac{1}{\cos x}$ for $x \neq \frac{\pi}{2} + k\pi, k \in \mathbb{Z}$;
- Cotangent: $\cot x = \frac{1}{\tan x}$ for $x \neq k\pi, k \in \mathbb{Z}$ (another symbol is $\operatorname{ctg} x$).

2 Formulas

This section contains some of most widely used trigonometric formulas.

1. Fundamental identity

$$\sin^2 x + \cos^2 x = 1, \qquad \forall x \in \mathbb{R}$$

2. Special angles and corresponding trigonometric values

$$\begin{array}{lll}
\sin\left(\frac{\pi}{6}\right) = \frac{1}{2} & \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2} & \sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} \\
\cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} & \cos\left(\frac{\pi}{3}\right) = \frac{1}{2} & \cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} \\
\tan\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{3} & \tan\left(\frac{\pi}{3}\right) = \sqrt{3} & \tan\left(\frac{\pi}{4}\right) = 1
\end{array}$$

3. Angles and symmetries

$\cos(\pi + x) = -\cos(x)$	$\sin(\pi + x) = -\sin(x)$	$\tan(\pi + x) = \tan(x)$
$\cos(-x) = \cos(2\pi - x) = \cos(x)$	$\sin(-x) = \sin(2\pi - x) = -\sin(x)$	$\tan(-x) = -\tan(x)$
$\cos(\pi - x) = -\cos(x)$	$\sin(\pi - x) = \sin(x)$	$\tan(\pi - x) = -\tan(x)$
$\cos(\frac{\pi}{2} + x) = -\sin(x)$	$\sin(\frac{\pi}{2} + x) = \cos(x)$	$\tan(\frac{\pi}{2} + x) = -\cot(x)$
$\cos(\frac{\pi}{2} - x) = \sin(x)$	$\sin(\frac{\pi}{2} - x) = \cos(x)$	$\tan(\frac{\pi}{2} - x) = -\cot(x)$

4. Addition formulas

$\cos(x + y) = \cos x \cos y - \sin x \sin y$	$\cos(x - y) = \cos x \cos y + \sin x \sin y$
$\sin(x + y) = \sin x \cos y + \sin y \cos x$	$\sin(x - y) = \sin x \cos y - \sin y \cos x$
$\tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$	$\tan(x - y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}$

5. Duplication formulas

$$\sin 2x = 2 \sin x \cos x, \qquad \cos 2x = \cos^2 x - \sin^2 x, \qquad \tan 2x = \frac{2 \tan x}{1 - \tan^2 x}.$$

6. Product-to-sum formulas

$$\sin x \sin y = \frac{1}{2}(\cos(x-y) - \cos(x+y)), \qquad \cos x \cos y = \frac{1}{2}(\cos(x+y) + \cos(x-y)),$$

$$\sin x \cos y = \frac{1}{2}(\sin(x + y) + \sin(x - y))$$

7. Sum-to-product formulas

$$\sin p + \sin q = 2 \sin \frac{p+q}{2} \cos \frac{p-q}{2}$$
$$\cos p + \cos q = 2 \cos \frac{p+q}{2} \cos \frac{p-q}{2}$$
$$\cos p - \cos q = -2 \sin \frac{p+q}{2} \cos \frac{p-q}{2}$$
$$\sin p - \sin q = 2 \cos \frac{p+q}{2} \sin \frac{p-q}{2}$$

Detailed examples

- Using the addition formulas, we write the following expressions in terms of $\sin x$, $\cos x$, $\tan x$

1. $\cos 6x$

$$\begin{aligned}
\cos 6x &= 2\cos^2 3x - 1 \\
&= 2(\cos(2x + x))^2 - 1 \\
&= 2(\cos 2x \cos x - \sin 2x \sin x)^2 - 1 \\
&= 2((2\cos^2 x - 1)\cos x - (2\sin x \cos x)\sin x)^2 - 1 \\
&= 2(2\cos^3 x - \cos x - 2(1 - \cos^2 x)\cos x)^2 - 1 \\
&= 2(2\cos^3 x - \cos x - 2\cos x + 2\cos^3 x)^2 - 1 \\
&= 2(4\cos^3 x - 3\cos x)^2 - 1 \\
&= 32\cos^6 x - 48\cos^4 x - 18\cos^2 x - 1.
\end{aligned}$$

2. $\sin\left(2x + \frac{\pi}{6}\right)$

$$\begin{aligned}
\sin\left(2x + \frac{\pi}{6}\right) &= \sin 2x \cos\frac{\pi}{6} + \cos 2x \sin\frac{\pi}{6} \\
&= \frac{\sqrt{3}}{2}\sin 2x + \frac{\sqrt{3}}{2}\cos 2x \\
&= \sqrt{3}\sin x \cos x + \frac{1}{2}(\cos^2 x - \sin^2 x)
\end{aligned}$$

3. $\tan 5x$

$$\tan 5x = \tan(x + 4x) = \frac{\tan 4x + \tan x}{1 - \tan x \tan 4x}$$

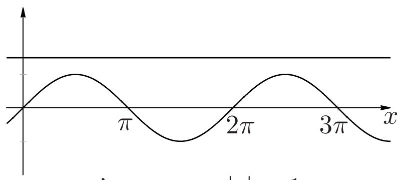
$$\begin{aligned}
\tan 4x &= \frac{2\tan 2x}{1 - \tan^2 2x} \\
&= \frac{2\frac{2\tan x}{1 - \tan^2 x}}{1 - \frac{4\tan^x}{(1 - \tan^2 x)^2}} \\
&= \frac{4\tan x(1 - \tan^2 x)}{\tan^4 x - 6\tan^2 x + 1}
\end{aligned}$$

$$\begin{aligned}
\frac{\tan 4x + \tan x}{1 - \tan x \tan 4x} &= \frac{\frac{4\tan x(1 - \tan^2 x)}{\tan^4 x - 6\tan^2 x + 1} + \tan x}{1 - \frac{4\tan x^2(1 - \tan^2 x)}{\tan^4 x - 6\tan^2 x + 1}} \\
&= \frac{\tan^5 x - 10\tan^3 x + 5\tan x}{5\tan^4 x - 10\tan^2 x + 1}
\end{aligned}$$

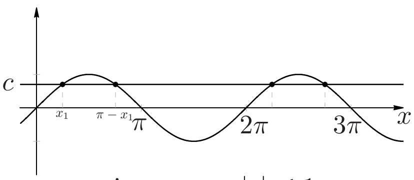
3 Trigonometric equations

3.1 Basic types

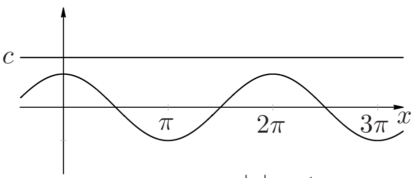
	$\sin x = c$	$\cos x = c$
$\lvert c\rvert > 1$	no solution	no solution
$\lvert c\rvert \leq 1$	infinitely many solutions, among which $x_1 \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and $x_2 = \pi - x_1$. Other solutions obtained by periodicity: $x_1 + 2k\pi$, $x_2 + 2k\pi$.	infinitely many solutions, among which $x_1 \in [0, \pi]$ and $x_2 = -x_1$. Other solutions obtained by periodicity: $x_1 + 2k\pi$, $x_2 + 2k\pi$.



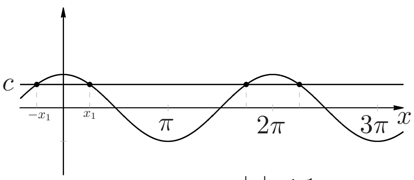
$\sin x = c, \quad |c| > 1$



$\sin x = c, \quad |c| \leq 1$



$\cos x = c, \quad |c| > 1$



$\cos x = c, \quad |c| \leq 1$

The equation $\tan x = c$ has exactly one solution x_1 in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$; all other solutions have the form $x_1 + k\pi, k \in \mathbb{Z}$.

3.2 Linear equations in sin and cos

Equations of the form $a \sin x + b \cos x = c$ can be solved by various means. In general, we can write the system

$$\begin{cases} a \sin x + b \cos x = c \\ \sin^2 x + \cos^2 x = 1 \end{cases}$$

Putting $\sin x = z$ and $\cos x = t$, we obtain

$$\begin{cases} az + bt = c \\ z^2 + t^2 = 1 \end{cases}$$

3.3 Homogeneous equations of degree 2 in sin, cos

Equations like

$$a \sin^2 x + b \cos^2 x + c \sin x \cos x = d$$

can be reduced to homogeneous ones by writing

$$a \sin^2 x + b \cos^2 x + c \sin x \cos x = d(\cos^2 x + \sin^2 x).$$

From this

$$(a - d) \sin^2 x + (b - d) \cos^2 x + c \sin x \cos x = 0.$$

If $a = d$ the solution is immediately found. Suppose then $a \neq d$ and note that the angles x such that $\cos x = 0$ do not give solutions; therefore we can divide by $\cos^2 x$ and get a quadratic equation in $t = \tan x$:

$$(a - d)t^2 + ct + (b - d) = 0.$$

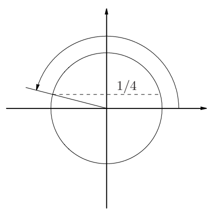
3.4 Broad indications to solve trigonometric equations

- Using the definitions write all trigonometric functions in terms of only sine and cosine.
- Write everything in terms of the same angle.
- In absence of source terms, factorize the equation and consider the factors separately.
- Examine carefully the admissible values when performing algebraic operations, such as dividing by a function.

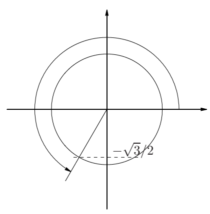
Detailed exercises

- Determine graphically the arc α such that

$\sin \alpha = \frac{1}{4}$ with $\frac{\pi}{2} < \alpha < \pi$

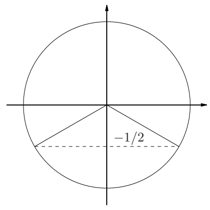


$\sin \alpha = -\frac{\sqrt{3}}{2}$ with $\pi < \alpha < \frac{3\pi}{2}$



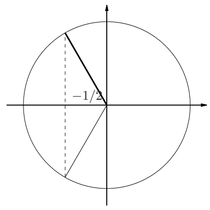
- Check on the picture how many angles satisfy the conditions

$\sin x = -\frac{1}{2}$ with $0 \leq x \leq \pi$



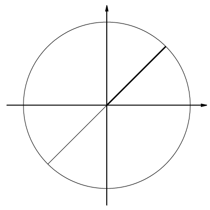
Answer: 0

$\cos x = -\frac{1}{2}$ with $0 \leq x \leq \pi$



Answer: 1

$\tan x = 1$ with $0 \leq x \leq \pi$



Answer: 1

- Solve the equations on $[0, 2\pi]$

1. $\sin\left(x + \frac{\pi}{4}\right) = \sin\left(2x + \frac{\pi}{3}\right)$

$$\sin\left(x + \frac{\pi}{4}\right) - \sin\left(2x + \frac{\pi}{3}\right) = 0$$

$$2\cos\left(\frac{x + 2x + \frac{\pi}{4} + \frac{\pi}{3}}{2}\right)\sin\left(\frac{x - 2x + \frac{\pi}{4} - \frac{\pi}{3}}{2}\right) = 0$$

$$2\cos\left(\frac{3}{2}x + \frac{7\pi}{24}\right)\sin\left(-\frac{1}{2}x - \frac{\pi}{24}\right) = 0$$

$$\begin{cases} \frac{3}{2}x + \frac{7\pi}{24} = \frac{\pi}{2} + k\pi \\ -\frac{1}{2}x - \frac{\pi}{24} = k\pi \end{cases}$$

Answer: $x = \frac{5}{36}\pi + \frac{2}{3}k\pi \cup x = -\frac{\pi}{12}\pi + 2k\pi$

2. $\cos^2 x + 2\cos x - 3 = 0$

$$t = \cos x$$

$$t^2 + 2t - 3 = 0$$

$$t = 1 \to \cos x = 1 \to x = 0$$

$$t = -3 \to \cos x = -3 \to \nexists x$$

Answer: $x = 0$

3. $\sqrt{\sin^2 x + \cos^2 x + \tan^2 x} = \frac{2}{\sqrt{3}}$

$$\sqrt{1 + \tan^2 x} = \frac{2}{\sqrt{3}}$$

$$\sqrt{1 + \frac{\sin^2 x}{\cos^2 x}} = \frac{2}{\sqrt{3}}$$

$$\sqrt{\frac{1}{\cos^2 x}} = \frac{2}{\sqrt{3}}$$

$$|\cos x| = \frac{\sqrt{3}}{2}$$

$$\cos x = \pm\frac{\sqrt{3}}{2}$$

Answer: $x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$

4 Trigonometric inequalities

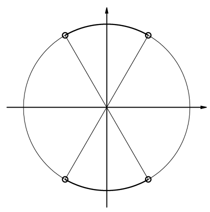
The procedure for solving a trigonometric inequality should always rely on the geometric viewpoint, by sketching the graphs of the maps involved, or looking at the unit circle.

For the basic inequalities $\sin x < c$, $\cos x < c$, $\tan x < c$, one solves first the corresponding equations, draws the map's graph and then determines the solution interval.

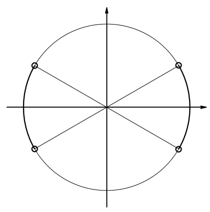
Examples

- Solve graphically

$|\tan x| > \sqrt{3}$

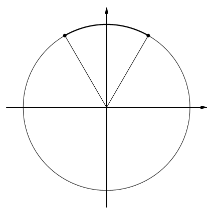


$|\sin x| < \frac{1}{2}$



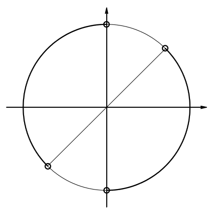
- Solve the inequalities on $[0, 2\pi]$

$2\sin x - \sqrt{3} \geq 0 \Rightarrow \sin x \geq \frac{\sqrt{3}}{2}$



Answer: $x \in [\frac{\pi}{3}, \frac{2\pi}{3}]$

$\tan x < 1$



Answer: $x \in [0, \frac{\pi}{4}) \cup (\frac{\pi}{2}, \frac{5\pi}{4}) \cup (\frac{3\pi}{2}, 2\pi]$

$\sin^2 x - \sin x \cos x \geq 0$

$$\sin x(\sin x - \cos x) \geq 0$$

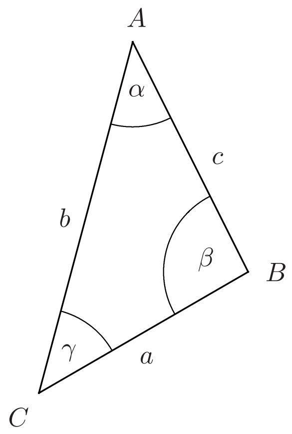
$$\begin{cases} \sin x \geq 0 \\ \sin x \geq \cos x \end{cases}$$

$$\begin{cases} \sin x \leq 0 \\ \sin x \leq \cos x \end{cases}$$

Answer: $x \in [\frac{\pi}{4}, \pi] \cup [\frac{5\pi}{4}, 2\pi]$

5 Properties of triangles

Let A, B, C be the vertices of a triangle, α, β, γ the angles at A, B, C and a, b, c the lengths of the sides opposite to A, B, C.



- Sine rule

$$\frac{a}{\sin \alpha} = \frac{b}{\sin \beta} = \frac{c}{\sin \gamma}$$

- Cosine rule

$$a^2 = b^2 + c^2 - 2bc \cos \alpha$$
$$b^2 = a^2 + c^2 - 2ac \cos \beta$$
$$c^2 = a^2 + b^2 - 2ab \cos \gamma$$

- Special case: if α is a right angle

$$b = a \sin \beta = a \cos \gamma, \quad c = a \sin \gamma = a \cos \beta, \quad b = c \tan \beta, \quad c = b \tan \gamma$$

6 EXERCISES – TRIGONOMETRY

EXERCISE 1

Using the fundamental identity and the addition formulas write:

1. $\sin 3x$ and $\sin 5x$ in terms of $\sin x$;
2. $\cos 3x$ and $\cos 4x$ in terms of $\cos x$;
3. $\tan 3x$ and $\tan 4x$ in terms of $\tan x$;
4. $(1 + \sin x)^2 - 2\sin x(1 + \cos x) + \sin 2x$ in terms of $\cos x$.

EXERCISE 2

Determine graphically the arc α such that:

1. $\sin\alpha = \frac{1}{3}$ with $0 < \alpha < \frac{\pi}{2}$
2. $\sin\alpha = -\frac{\sqrt{2}}{2}$ with $-\frac{\pi}{2} < \alpha < 0$
3. $\sin\alpha = -\frac{3}{5}$ with $-\pi < \alpha < -\frac{\pi}{2}$
4. $\cos\alpha = -\frac{1}{2}$ with $\frac{\pi}{2} < \alpha < \pi$
5. $\cos\alpha = -\frac{1}{6}$ with $\pi < \alpha < 2\pi$
6. $\cos\alpha = -\frac{\sqrt{3}}{2}$ with $\pi < \alpha < \frac{3\pi}{2}$

EXERCISE 3

Solve graphically (using the unit circle):

1. $|\tan x| > 1$
2. $|\sin x| \leq \frac{\sqrt{3}}{2}$

EXERCISE 4

Check graphically how many angles satisfy the following equations

1. $\sin x = \frac{1}{2}$ with $0 \leq x \leq 2\pi$
2. $\tan x = \frac{1}{3}$ with $0 \leq x \leq \frac{\pi}{2}$
3. $\cos x = -\frac{1}{5}$ with $0 \leq x \leq \frac{\pi}{2}$
4. $\cos x = \frac{7}{6}$ with $0 \leq x \leq 2\pi$
5. $\sin x = -\frac{1}{8}$ with $0 \leq x \leq \pi$
6. $\sin x = -\frac{1}{8}$ with $0 \leq x \leq \frac{3\pi}{2}$
7. $\tan x = 5$ with $0 \leq x \leq 2\pi$

EXERCISE 5

1. The solutions to $\sin x = \cos x$ are:
   (a) $x = 2k\pi$ for any integer k
   (b) $x = \frac{\pi}{4} + k\pi$ for any integer k
   (c) $x = \frac{\pi}{4} + 2k\pi$ for any integer k
   (d) $x = \pi + 2k\pi$ for any integer k
   (e) none of the above answers is correct.

2. The solutions to $\sin^2 x + \cos^2 x > 1$, on $[0, 2\pi]$, are:
   (a) $0 < x < \frac{\pi}{2}$
   (b) $\frac{\pi}{4} < x < \frac{3\pi}{4}$
   (c) no x
   (d) any x
   (e) none of the above answers is correct.

EXERCISE 6

Solve the following equations on $0 \leq x < 2\pi$:

1. $\sqrt{2} \sin x + 1 = 0$
2. $\cos x = -\frac{1}{2}$
3. $\tan x = 1$
4. $2 \sin x - \sqrt{3} = 0$
5. $\sqrt{2} \cos x - 1 = 0$
6. $2 \cos^2 x - 1 = 0$
7. $3 \tan^2 x - 1 = 0$
8. $|\sin x| = 1$
9. $\sin 2x = \frac{1}{2}$
10. $\cos 3x = -\frac{1}{2}$
11. $\cos(2x + \frac{\pi}{4}) = \frac{\sqrt{2}}{2}$
12. $\sin(x + \frac{\pi}{3}) = \sin(2x)$
13. $\sin(x + \frac{\pi}{6}) = \sin(2x + \frac{3\pi}{4})$
14. $\tan(\frac{\pi}{6} - 2x) = \tan(3x - \frac{\pi}{3})$
15. $\sqrt{\tan^2 x + 1} = 2$
16. $\sin x - \cos x = 0$
17. $\cos x + \sin x \cos x = 0$
18. $\sin x - 2 \sin x \cos x = 0$
19. $2 \cos^2 x - \cos x - 1 = 0$
20. $2 \sin^2 x - 5 \sin x + 2 = 0$

EXERCISE 7

Solve the inequalities on $[0, 2\pi]$:

1. $2 \sin x - 1 \leq 0$
2. $\cos x < -\frac{1}{2}$
3. $\tan x < \frac{\sqrt{3}}{3}$
4. $2 \cos(x - \frac{\pi}{3}) - 1 < 0$
5. $\sin x + 2 \sin^2 x < 1$
6. $\sin x + \cos x \geq 0$
7. $\sin x - \cos x > 0$
8. $3 \tan^2 x - 1 < 0$ in $(0, \pi)$
9. $\sqrt[3]{\tan(x - \frac{\pi}{4})} < 1$
10. $2 \cos^2 x + \cos x > 0$

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11. $2\cos^2 x + \cos x - 1 > 0$
12. $2\cos^2 x - 5\cos x - 3 > 0$
13. $2^{\sin x - \cos x} > 1$

Exercise 8

Given the following two elements of a right triangle ABC with hypotenuse AB, determine the remaining angles and sides:

1. $\overline{AB} = 4$ and $\beta = 30°$
2. $\overline{AB} = 6$ and $\alpha = 45°$
3. $\overline{BC} = \sqrt{3}$ and $\alpha = 60°$
4. $\overline{AC} = 7$ and $\beta = 75°$

Exercise 9

Given the following three elements of a generic triangle, determine the remaining angles and sides:

1. $\overline{AB} = \overline{AC} = 5$ and $\alpha = 30°$
2. $\overline{AB} = \sqrt{3}$; $\overline{\mathrm{BC}} = 2$ and $\beta = 60°$
3. $\overline{BC} = 7$, $\overline{\mathrm{AC}} = 3$ and $\alpha = 45°$
4. $\overline{AC} = 2$, $\alpha = 30°$, $\beta = 75°$

7 SOLUTIONS

EXERCISE 1

1. $\sin 3x = -4\sin^3 x + 3\sin x$, $\sin 5x = 15\sin^5 x - 20\sin^3 x + 5\sin x$;
2. $\sin 3x = 4\cos^3 x - 3\cos x$, $\cos 4x = 8\cos^4 x - 8\cos^2 x + 1$;
3. $\tan 3x = \frac{3\tan x - \tan^3 x}{1 - 3\tan^2 x}$, $\tan 4x = \frac{7\tan x(1-\tan^x)}{1+\tan^4 x - 6\tan^2 x}$;
4. $(1+\sin x)^2 - 2\sin x(1+\cos x) + \sin 2x = 2 - \cos^2 x$.

EXERCISE 4

1. 2
2. 1
3. none
4. none
5. none
6. 1
7. 2

EXERCISE 5

Solutions: b, c.

EXERCISE 6

1. $\frac{5\pi}{4}, \frac{7\pi}{4}$
2. $\frac{2\pi}{3}, \frac{4\pi}{3}$
3. $\frac{\pi}{4}, \frac{5\pi}{4}$
4. $\frac{\pi}{3}, \frac{2\pi}{3}$
5. $\frac{\pi}{4}, \frac{7\pi}{4}$
6. $\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$
7. $\frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$
8. $\frac{\pi}{2}, \frac{3\pi}{2}$
9. $\frac{\pi}{12}, \frac{5\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}$
10. $\frac{2\pi}{9}, \frac{4\pi}{9}, \frac{10\pi}{9}, \frac{14\pi}{9}, \frac{16\pi}{9}, \frac{8\pi}{9}$
11. $0, \frac{3\pi}{4}, \pi, \frac{7\pi}{4}$
12. $\frac{\pi}{3}, \frac{2\pi}{9}, \frac{14\pi}{9}, \frac{8\pi}{9}$
13. $\frac{\pi}{36}, \frac{17\pi}{12}, \frac{25\pi}{36}, \frac{49\pi}{36}$
14. $\frac{\pi}{10} + k\frac{\pi}{5}$
15. $\frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$
16. $\frac{\pi}{4}, \frac{5\pi}{4}$
17. $\frac{\pi}{2}, \frac{3\pi}{2}$
18. $0, \pi, \frac{\pi}{3}, \frac{5\pi}{3}$
19. $0, \frac{2\pi}{3}, \frac{4\pi}{3}$
20. $\frac{\pi}{6}, \frac{5\pi}{6}$

EXERCISE 7

1. $\left[0, \frac{\pi}{6}\right] \cup \left[\frac{5\pi}{6}, 2\pi\right]$
2. $\left(\frac{2\pi}{3}, \frac{4\pi}{3}\right)$
3. $\left(0, \frac{\pi}{6}\right) \cup \left(\frac{\pi}{2}, \frac{7\pi}{6}\right) \cup \left(\frac{3\pi}{2}, 2\pi\right)$
4. $\left(\frac{2\pi}{3}, 2\pi\right)$
5. $\left[0, \frac{\pi}{6}\right) \cup \left(\frac{5\pi}{6}, 2\pi\right] \setminus \left\{\frac{3}{2}\pi\right\}$
6. $\left[0, \frac{3\pi}{4}\right] \cup \left[\frac{7\pi}{4}, 2\pi\right)$
7. $\left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$
8. $\left[0, \frac{\pi}{6}\right) \cup \left(\frac{5\pi}{6}, \pi\right]$
9. $\left[\frac{\pi}{4}, \frac{\pi}{2}\right] \cup \left(\frac{3\pi}{4}, \frac{3\pi}{2}\right) \cup \left(\frac{7\pi}{4}, 2\pi\right]$
10. $\left(\frac{2\pi}{3}, \frac{4\pi}{3}\right) \cup \left[0, \frac{\pi}{2}\right) \cup \left(\frac{3\pi}{2}, 2\pi\right]$
11. $\left[0, \frac{\pi}{3}\right] \cup \left[\frac{5\pi}{3}, 2\pi\right]$
12. $\left(\frac{2\pi}{3}, \frac{4\pi}{3}\right)$

Exercise 8

1. $\alpha = 60°; \overline{AC} = 4; \overline{BC} = 2\sqrt{3};$
2. $\beta = 45°; \overline{AC} = \overline{BC} = 3\sqrt{2};$
3. $\beta = 30°; \overline{AC} = 1; \overline{AB} = 2;$
4. $\alpha = 15°; \overline{AB} = 14\sqrt{2-\sqrt{3}}; \overline{BC} = 7(2-\sqrt{3})$

Exercise 9

1. $\beta = \gamma = 75°; \overline{BC} = 5\sqrt{2-\sqrt{3}}$
2. $\alpha = 67.08°; \gamma = 52.9°; \overline{AC} = \sqrt{7-2\sqrt{3}}$
3. $\beta = 17.64°; \gamma = 117.36°; \overline{AB} = 8.79$
4. $\gamma = 75°; \overline{AB} = 2; \overline{BC} = 2\sqrt{2-\sqrt{3}}$