Politecnico di Torino

Bridging course in mathematics

Sheet 4

Exponential and logarithms

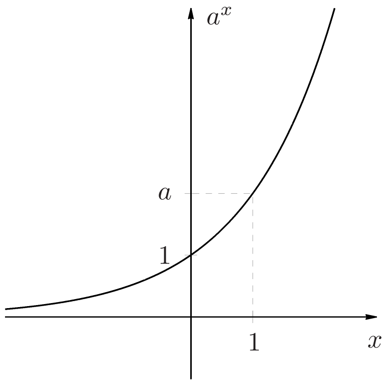
1 The exponential map

Recall that for every $a > 0$ and any $x, y \in \mathbb{R}$ the following properties hold

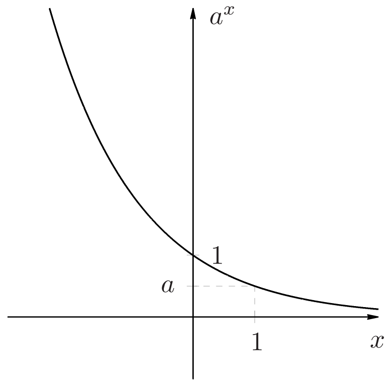
$$a^0 = 1, \qquad a^1 = a, \qquad a^{-1} = \frac{1}{a}, \qquad a^{x+y} = a^x a^y, \qquad (a^x)^y = a^{xy}.$$

Given a real number $a > 0$, one calls *exponential map in base* a the function $x \mapsto a^x$.

- If $a = 1$, the map is constant, in fact $1^x = 1$ for any real x.
- If $a \neq 1$ its domain is $\mathbb{R}$ and its range $(0, +\infty)$.
  - If $a > 1$, the function is increasing (see below, left);
  - If $a < 1$, the function is decreasing (see below, right).



$a > 1$



$a < 1$

- The graphs of the exponential maps in base $a \neq 1$ and $1/a$ are symmetric to each other with respect to the y-axis. In fact

$$y(x) = \left(\frac{1}{a}\right)^x = a^{-x}\ .$$

Remark 1.1 *Among all exponential functions a special role is played by the map in base* e. *The precise definition and features of the number* e *will be discussed in the Calculus course. For the time being, note that since* $\mathrm{e} \approx 2,718$, *the exponential in base* e *has an intermediate behaviour between the maps* $y = 2^x$ *and* $y = 3^x$.

2 The logarithm function

Given the function $f(x) = a^x$, with a a positive real number, and the positive real y_0, consider the equation $a^x = y_0$.

- If $a = 1$ the equation is solved by any real number if $y_0 = 1$, while it has no solution for $y_0 \neq 1$.
- If $a \neq 1$, for any given $y_0 > 0$ the equation has one, and only one, solution x_0.

The real number x_0 is said the logarithm in base a of the positive real number y_0: $x_0 = \log_a y_0$. Varying $y_0 \in (0, \infty)$ defines a map called *logarithm in base* a: $x \mapsto \log_a x$.

Its domain is $(0, +\infty)$, its range $\mathbb{R}$.

The exponential and the logarithm satisfy these properties

$$a^{\log_a y_0} = y_0 \text{ for any } y_0 \in (0, +\infty),$$
$$\log_a (a^{x_0}) = x_0 \text{ for any } x_0 \in \mathbb{R}.$$

2.1 Properties of the logarithm

Consider positive real numbers a, x and y, with $a \neq 1$; let z be another given real. Then:

- $\log_a xy = \log_a x + \log_a y$
- $\log_a \frac{x}{y} = \log_a x - \log_a y$
- $\log_a x^z = z \log_a x$.

Furthermore, if b is a positive real, $b \neq 1$, then the formula of *base change* for logarithms holds:

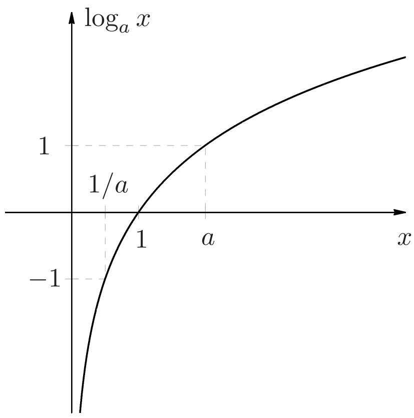
$$\log_b x = \frac{\log_a x}{\log_a b}.$$

2.2 Graph of the logarithm

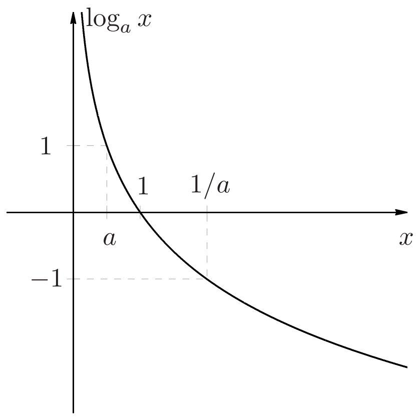
Thanks to the relationship between logarithms and exponentials, if the point (p, q) belongs to the exponential graph, then (q, p) lies on the logarithm graph.

The graphs of the exponential in base a (assuming $a > 0$ and $a \neq 1$) is the mirror-symmetric to the graph of the logarithmic map in the same base with respect to the bisectrix of the first and third quadrants.

The left figure shows the logarithm in base $a > 1$, while on the right we have $a < 1$.



$a > 1$



$a < 1$

Let's see some properties of the function $y = \log_a x$:

1. the graph contains the points $(1, 0), (a, 1), \left(\frac{1}{a}, -1\right)$;
2. if $a > 1$ the map is increasing, negative on $(0, 1)$ and positive on $(1, \infty)$;
3. if $a < 1$ the map is decreasing, positive on $(0, 1)$ and negative on $(1, \infty)$.

Remark 2.1 *Choosing* e *as base defines the so-called* natural logarithm*, customarily denoted by* $y = \ln x$*.*

3 Exponential and logarithmic equations

In the following tables we list some types of exponential and logarithmic equations/inequalities (left column) and a solving technique (right column).

3.1 Exponential equations

$a^x = k$ with $a > 0,\ a \neq 1$ and $k \in \mathbb{R}$	if $k > 0$, $x = \log_a k$.
$a^{f(x)} = a^{g(x)}$	$f(x) = g(x)$
$a^{f(x)} = b^{g(x)}, \quad b > 0, \quad b \neq 1$	set $b^{g(x)} = a^{g(x)\log_a b}$
$f(a^x) = 0$	set $a^x = t$ and solve $f(t) = 0$

3.2 Exponential inequalities

$a^{f(x)} > a^{g(x)}, \quad a > 0, \quad a \neq 1$	if $a > 1,\ f(x) > g(x)$ if $a < 1,\ f(x) < g(x)$
$f(a^x) > c$	set $a^x = t$ and solve $f(t) > c$

3.3 Detailed examples

1. Let's solve $2^{2x^2+x} - 2^{x^3+2x} = 0$.

   We can rewrite the equation in the form $2^{2x^2+x} = 2^{x^3+2x}$; this equality implies that the two exponents are equal (by the exponential map's properties); hence

$$\begin{aligned} x(2x+1) = x(x^2+2) &\implies x(-x^2+2x-1) = 0 \\ &\implies -x(x-1)^2 = 0 \implies x = 0 \text{ or } x = 1 \ . \end{aligned}$$

2. We solve $\dfrac{2^{x+1}5^{x-1}}{3^x} = 2$.

   Multiply both sides by 3^x (which is always positive) to get

$$2^{x+1}5^{x-1} = 2 \cdot 3^x \implies 2^x 5^{x-1} = 3^x$$

   Now take the logarithm (for instance, in base e), and recalling its properties,

$$\begin{aligned} \ln 2^x + \ln 5^{x-1} = \ln 3^x &\implies x \ln 2 + x \ln 5 - x \ln 3 = \ln 5 \\ &\implies x = \frac{\ln 5}{\ln 2 + \ln 5 - \ln 3} \ . \end{aligned}$$

   If we had chosen the base 10, going over the same reasoning we would have found $x = \dfrac{\log_{10} 5}{\log_{10} 2 + \log_{10} 5 - \log_{10} 3}$; does this coincide with what we have found above?

3. Let's solve $\left(\left(\frac{1}{7}\right)^{x+1}\right)^x > \frac{1}{49}$.

   We write it as

$$\left(\frac{1}{7}\right)^{(x+1)x} > \left(\frac{1}{7}\right)^2 \ .$$

   The exponential in base 1/7 decreases, so the inequality implies that $(x+1)x < 2$. Hence $(x-1)(x+2) < 0$, which gives $-2 < x < 1$.

4. Solve $4^x - 2 \cdot 2^x - 3 \leq 0$.

   This is equivalent to $2^{2x} - 2 \cdot 2^x - 3 \leq 0$; substituting $t = 2^x$ we are lead to the quadratic inequality $t^2 - 2t - 3 \leq 0$, solved by $-1 \leq t \leq 3$.

   Going back to the variable x, we have $-1 \leq 2^x \leq 3$. The first inequality $-1 \leq 2^x$ holds for any x, whilst $2^x \leq 3$ tells that $x \leq \log_2 3$.

3.4 Logarithmic equations

$\log_a x = b$ with $a > 0,\ a \neq 1$ and $b \in \mathbb{R}$	$x = a^b$
$\log_a f(x) = b$ with $a > 0,\ a \neq 1, b \in \mathbf{R}$.	if $f(x) > 0$, $f(x) = a^b$
$\log_a f(x) = \log_a g(x)$	if $f(x) > 0$ and $g(x) > 0$, $f(x) = g(x)$
$f(\log_a x) = 0$	set $\log_a x = t$ and solve $f(t) = 0$

3.5 Logarithmic inequalities

$\log_a f(x) > \log_a g(x)$	for $a > 1,\ f(x) > g(x)$ for $a < 1,\ f(x) < g(x)$
$f(\log_a x) > c$	set $\log_a x = t$ then solve $f(t) > c$

3.6 Explicit examples

1. We want to solve $\log_4{(x+6)} + \log_4 x = 2$.

   The first thing is to observe that the functions are defined for $x + 6 > 0$ and $x > 0$; therefore the equation is defined over the intersection of these two sets, ie $D = (0, +\infty)$.

   Writing $\log_4{\left(x^2 + 6x\right)} = 2$, and exponentiating in base 4, we obtain $x^2 + 6x = 16 \Longrightarrow x^2 + 6x - 16 = 0$, solved by $x_1 = -8$ and $x_2 = 2$.

   The solution x_1 is not valid (it doesn't belong to D), whereas $x_2 \in D$; thus the given equation has only one solution $x_2 = 2$.

2. Solve $\log_2{(x+1)} = \log_4{(2x+5)}$.

   Here as well we must pay attention to the logarithms' domains; the left logarithm is defined for $x > -1$, the other one for $x > -5/2$. The equation then makes sense on $D = (-1, +\infty)$.

   We have logarithms in two different bases, so we need to uniformize (to base two, say) using the formula for base change on the right-hand side:

   $$\log_2{(x+1)} = \frac{\log_2{(2x+5)}}{\log_2 4} \ .$$

   Using now known properties,

   $$\begin{aligned} \log_2{(x+1)} = \tfrac{1}{2}\log_2{(2x+5)} \quad &\Longrightarrow \log_2{(x+1)} = \log_2{(2x+5)}^{\frac{1}{2}} \\ &\Longrightarrow x + 1 = \sqrt{2x+5} \ . \end{aligned}$$

   Now we square to get $x^2 - 4 = 0$, solved by $x_1 = -2$, not valid because not in D, and $x_2 = 2$, which is therefore the only solution.

3. Solve $\log_2 x - \log_2 3 < \log_2(x+2)$.

   Begin with noting that $x \in D = (0, \infty)$ necessarily. Let's write the left-hand side as one logarithm

   $$\log_2 x - \log_2 3 = \log_2 \frac{x}{3} < \log_2(x+2) \ .$$

   Since the log in base 2 is strictly increasing, we have $\frac{x}{3} < x + 2$ hence $x > -\frac{1}{3}$. Keeping in mind the existence domain, the result is that the solution set is D itself.

4. Let's solve $\log_2^3 x - 2\log_2 x > 0$.

   In this case, too, $x \in D = (0, +\infty)$. Setting $t = \log_2 x$ yields the inequality $t^3 - 2t = t(t^2 - 2) > 0$. The straight line $y = t$ and the parabola $y = t^2 - 2$ have the same sign if $t > \sqrt{2}$ or $-\sqrt{2} < t < 0$ (check using graphs).

   Back to the variable x we have $\log_2 x > \sqrt{2}$, which holds on $D_1 = \left(2^{\sqrt{2}}, +\infty\right)$, and also $-\sqrt{2} < \log_2 x < 0$, valid on $D_2 = \left(2^{-\sqrt{2}}, 1\right)$.

   These are intervals contained in D, so the solutions set is $D_1 \cup D_2$.

4 EXERCISES – EXPONENTIALS AND LOGARITHMS

True or false?

1. $\ln x < \ln(x + 10)$ **T** **F**
2. $\log_5(0,2) = -\frac{1}{2}$ **T** **F**
3. $\log_3(0,3) < 0$ **T** **F**
4. $\ln(x - 1) < 1 \Leftrightarrow x - 1 < \mathrm{e}$ **T** **F**
5. $\log_a 9 = 3 \log_a 2$ **T** **F**
6. $\ln(\sqrt{7}) = \frac{1}{2} \ln 7$ **T** **F**
7. The graph of $y = \ln(x - 2)$ and that of $y = \ln x$ can be obtained from one another by translation. **T** **F**
8. $\mathrm{e}^x > \ln x$ for any positive real x **T** **F**
9. $4^n(2^n + 8^n) - 2^{3n} = 2^{5n}$ **T** **F**
10. $\dfrac{2^x}{2^{x^2-1} + 2^{x-1}} = 2^{2+x} \quad \forall x \in \mathbb{R}$ **T** **F**
11. $2^6 = 8^x \Leftrightarrow x = 2$ **T** **F**
12. $3^x = \frac{1}{9} \Leftrightarrow x = 2$ **T** **F**

Exercise 1

1. The solution to $\log_{10}(10^x) = 5$ is:
   (a) $x = 10^5$
   (b) $x = 0$
   (c) $x = 5$
   (d) $x = \log_{10}(5)$
   (e) none of the above answers is correct
2. The equation $\mathrm{e}^x = -x$ has:
   (a) no real solution
   (b) one real solution
   (c) two real solutions
   (d) infinitely many real solutions
   (e) none of the above is correct
3. The solutions to $\frac{3^{x^2}}{3} = 1$ are:
   (a) $x = 0$
   (b) $x = 1$
   (c) $x = 1$ and $x = -1$
   (d) no real number
   (e) none of the above is correct
4. The expression $-\log(x)$ equals:
   (a) $\frac{1}{\log x}$
   (b) $\log(-x)$
   (c) $\log(1 - x)$
   (d) $\log(-\frac{1}{x})$

(e) none of the above is correct

5. The solutions to $e^{x+1} > \frac{1}{e^{-x}}$ are:
   (a) no real x
   (b) any real $x > 0$
   (c) any real $x > 1$
   (d) any real x
   (e) none of the above is correct.

EXERCISE 2

Determine the domain of the maps:

1. $y(x) = \log_2 2x^4$
2. $y(x) = \ln 3x\,(5-x)^2$
3. $y(x) = \ln \frac{x-1}{x+2}$
4. $y(x) = \ln\left(x^3 - 1\right)$
5. $y(x) = \log_5\left(3\,|x| - x^2 - 7\right)$

EXERCISE 3

Imposing suitable restrictions, solve:

1. $(3^x)^{x+3} = 1$
2. $3 \cdot 2^{x+1} - 2^{x-1} - 5 \cdot 2^x = 8$
3. $5^x - 4 = 5^{1-x}$
4. $2^{3x+1} + 2^{3x+2} - 3 \cdot 8^x = 32$
5. $7^{2x} - 5 \cdot 7^x + 6 < 0$
6. $\left(\frac{1}{3}\right)^{x+2} > 1$
7. $5^{x^2-x} > 0$
8. $\log_{10}(x+5) = 1$
9. $\log_2 \dfrac{x-1}{x+1} = \log_2 x$
10. $\log_a^2 x - 2\log_a x + 1 = 0$
11. $\log_2(\log_3(6x+1)) = 0$
12. $\dfrac{\log_3(x+2) - 1}{2 + \log_3 x} = 1$
13. $\dfrac{\log_2(1-x)}{\log_2(x^2+x)} = \dfrac{1}{3}$
14. $\log(9-x) < 0$
15. $\log_3(x^2+1) > 0$
16. $\log_2 x - 3 > -\dfrac{2}{\log_2 x}$
17. $\dfrac{e^{2x} - 8e^x + 7}{e^x - 4} < 0$
18. $\dfrac{2e^{-x} + e^x - 3}{1 - e^x} > 0$

EXERCISE 4

Solve algebraically the following equations and inequalities, providing a geometrical explanation:

1. $e^{(|x|)} - 3 < 0$
2. $\log_3(x-8) > 0$
3. $|e^x - 2| = 1$
4. $\log_3 |x| = 2$

Exercise 5

Solve graphically the following equations, discussing the existence and the number of solutions as the real number k varies. The idea is that you should get an idea of how things go by sketching the graphs for some values of k.

1. $e^{-|x+2|} = k$
2. $2^x = k - x$
3. $e^x = kx$
4. $|\ln|x|| = k$
5. $\ln x = k - x$
6. $\log_2 x = x - k$

EXERCISE 6

Find a polynomial $P(x)$ such that $\log P(x)$ doesn't define a function. Does there exist a polynomial $P(x)$ for which $\sqrt{P(x)}$, $\sqrt[3]{P(x)}$, and $e^{P(x)}$ don't exist?

5 SOLUTIONS

True or false?

1. T
2. F
3. T
4. T
5. F
6. T
7. T
8. T
9. T
10. F
11. T
12. F

Exercise 1

Correct answers:

1. c,
2. b,
3. c,
4. e,
5. d

Exercise 2

1. $x \neq 0$
2. $x > 0$ $x \neq 5$
3. $x < -2$ $x > 1$
4. $x > 1$
5. no x

Exercise 3

1. $\{0, -3\}$
2. $x = 4$
3. $x = 1$
4. $\frac{\ln 5 - \ln 3}{\ln 2 - \ln 3}$
5. $\log_7 2 < x < \log_7 3$
6. $x < -2$
7. $\forall x \in \mathbb{R}$
8. $x = 5$
9. no x
10. $x = a$
11. $x = \frac{1}{3}$
12. $x = \frac{1}{13}$
13. $x = \frac{1}{3}$
14. $8 < x < 9$
15. $x \neq 0$
16. $1 < x < 2, \; x > 4$
17. $x < 0, \; \ln 4 < x < \ln 7$
18. $x < \ln 2, \; x \neq 0$

Exercise 4

1. $-\ln 3 < x < \ln 3$
2. $x > 9$
3. $x = 0, \; x = \ln 3$
4. $x = \pm 9$