Politecnico di Torino

Bridging course in mathematics

Sheet 3

Rational and irrational equalities and inequalities

1 Equations

Determining *the zero set* of a function f, that is, finding the elements x of the domain where f becomes zero, means finding the solutions to the equation $f(x) = 0$ in the variable x.

Determining the *level sets*, the elements in the domain where the map assumes a given value k, means solving the equation $f(x) = k$, ie $f(x) - k = 0$.

More generally, the problem is of the following kind: given two maps f, g, determine the elements x such that $f(x) = g(x)$, or $f(x) - g(x) = 0$.

Definition 1.1 *Two equations are called* ***equivalent*** *if every solution to the former is a solution to the latter, and conversely, if every solution to the latter also solves the former.*

An equation transforms into an equivalent one by:

- adding or subtracting to both sides the same function defined over $\mathbb{R}$;
- multiplying or dividing both sides by the same function defined over $\mathbb{R}$ and non-zero.

Warning: if we add a function $h(x)$ that is not defined everywhere on $\mathbb{R}$, the new equation might no longer be equivalent to the original one. Analogous problems may arise with the multiplication.

For example: the equation $x^2 = 4$ has two solutions $x = 2$ and $x = -2$; by multiplying by $\frac{1}{x-2}$ we lose the solution $x = +2$.

2 Inequalities

Determining where a map is positive, that is to say, finding the subset of the domain where $f(x) > 0$, means solving the corresponding inequality.

In the same way one may find the set where the map is negative $f(x) < 0$, non-negative $f(x) \geq 0$, or non-positive $f(x) \leq 0$.

Similarly to what we have seen for equalities, we may consider inequalities of the type $f(x) < g(x)$ or $f(x) > g(x)$: the latter allows to determine the domain elements for which the graph of f lies "above" the graph of g.

An inequality transforms into an equivalent one by adding or subtracting to both sides the same function defined over $\mathbb{R}$, or by multiplying or dividing both sides by the same function defined over $\mathbb{R}$ and strictly positive.

Multiplying the inequality $f(x) > g(x)$ by a map $h(x)$ that is defined over $\mathbb{R}$ and negative has the effect of producing the equivalent inequality $f(x)h(x) < g(x)h(x)$.

3 Equations and inequalities of degree 1

An equation of degree 1 has the form

$$ax + b = 0,$$

where a and b are real numbers.

For $a = 0$ we have two cases:

- if $b \neq 0$, the equation has no solution and is called *inconsistent*;
- if $b = 0$, the equation has infinitely many solutions, and is called *indeterminate*.

For $a \neq 0$ the solution is unique, namely $x = -\frac{b}{a}$.

An inequality of degree 1 has the form

$$ax + b > 0,$$

where a and b are real.

Excluding the trivial case $a = 0$, to study the inequality we have to distinguish the cases $a > 0$ and $a < 0$:

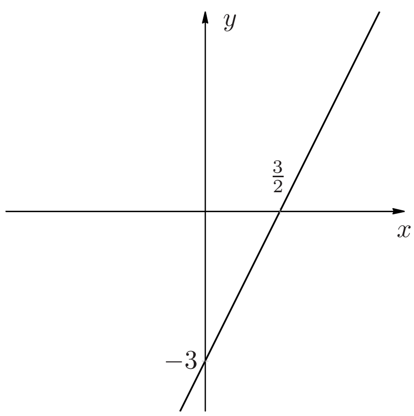
- if $a > 0$ the solutions are given by the set of $x > -\frac{b}{a}$;
- if $a < 0$ the solutions are given by the set of $x < -\frac{b}{a}$.

Example

The results obtained for the following equality and inequality are confirmed by geometrical considerations:

$$\begin{aligned} 2x - 3 = 0 \quad &\Rightarrow \quad x = \tfrac{3}{2} \\ 2x - 3 > 0 \quad &\Rightarrow \quad x > \tfrac{3}{2} \end{aligned}$$

The graph intersects the x-axis at $x = \frac{3}{2}$ and is positive for $x > \frac{3}{2}$.



4 Equations and inequalities of degree 2

An equation of degree 2 (a quadratic equation) is of the form

$$ax^2 + bx + c = 0,$$

where a, b, c are real numbers. When $a = 0$ the equation reduces to degree one, so from now on we shall assume $a \neq 0$. Let's consider a few special cases:

- if $c = 0$ the equation reads
$$ax^2 + bx = 0 \Rightarrow x(ax + b) = 0,$$
which is solved by $x_1 = 0$ and $x_2 = -b/a$;
- if $b = 0$ the equation reads
$$ax^2 + c = 0 \Rightarrow x^2 = -\frac{c}{a},$$
solved by $x_{1,2} = \pm\sqrt{-\frac{c}{a}}$ provided $-\frac{c}{a} \geq 0$.

In general, to solve second-degree equations one has to compute the discriminant

$$\Delta = b^2 - 4ac.$$

Then

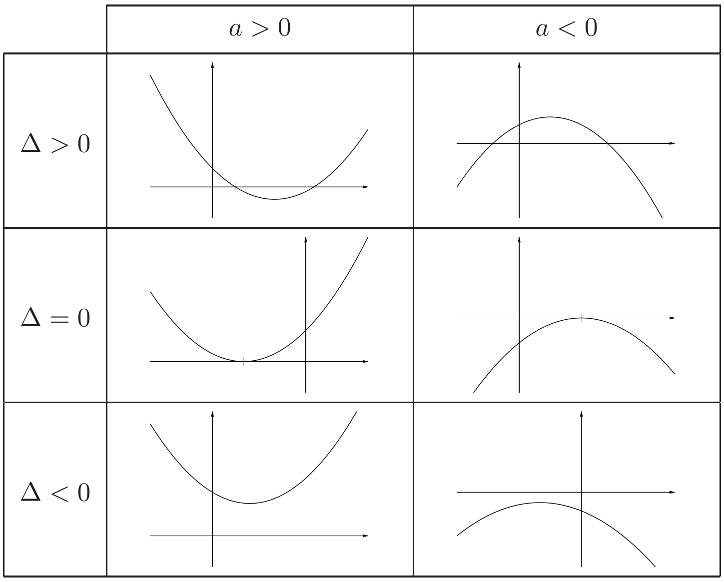
- if $\Delta > 0$ the equation has two distinct real solutions: $x_{1,2} = \frac{-b \pm \sqrt{\Delta}}{2a}$;
- if $\Delta = 0$ the equation has two coinciding real solutions: $x_1 = x_2 = -\frac{b}{2a}$;
- if $\Delta < 0$ the equation has no real solutions.

Remark 4.1 *In case $\Delta \geq 0$ the following relationships hold between the solutions x_1 and x_2 and the equation's coefficients:*

$$\begin{cases} x_1 \cdot x_2 = c/a \\ x_1 + x_2 = -b/a \end{cases}$$

4.1 Inequalities of degree 2

From the previous sheet we know the vertex of the parabola $y = ax^2 + bx + c$ is $V = (-\frac{b}{2a}; -\frac{\Delta}{4a})$, and if $a > 0$ the graph is U-shaped (convex), while for $a < 0$ it is concave.



The study of equations has taught us that $\Delta > 0$ forces the parabola to have two zeroes (two intersections with the x-axis), $\Delta = 0$ implies the existence of one double zero (the parabola is tangent to the axis at its vertex), and $\Delta < 0$ says the parabola doesn't meet the x-axis.

These considerations hint at a general method for solving inequalities of degree two like

$$ax^2 + bx + c > 0.$$

Given the solutions x_1 and x_2 to the corresponding equality, we have

- if $a > 0$ and $\Delta > 0$ the solutions are given by any $x \in (-\infty, x_1) \cup (x_2, +\infty)$;
- if $a > 0$ and $\Delta = 0$ the solutions are given by the set $\mathbb{R} \setminus \{-b/2a\}$;
- if $a > 0$ and $\Delta < 0$ the solutions are given by any $x \in \mathbb{R}$;

- if $a < 0$ and $\Delta > 0$ the solutions are given by the interval (x_1, x_2);
- if $a < 0$ and $\Delta = 0$ or $\Delta < 0$ there are no solutions.

The other inequality signs are dealt with in a similar way.

5 Fractional equations and inequalities

Consider the equation

$$\frac{f(x)}{g(x)} = 0$$

defined where $g(x) \neq 0$.

To find its solutions it suffices to solve $f(x) = 0$, possibly excluding the solutions such that $g(x)$ is zero.

Recall that a fraction is positive when numerator and denominator are both positive or both negative. Then, to solve

$$\frac{f(x)}{g(x)} > 0$$

we must solve the systems

$$\begin{cases} f(x) > 0 \\ g(x) > 0 \end{cases} \qquad \begin{cases} f(x) < 0 \\ g(x) < 0 \end{cases}$$

and take the union of the solution sets. The other signs are treated similarly.

5.1 Detailed examples

1. Let's solve $\dfrac{x^2-1}{x-3} \geq 0$.

   First of all we solve the equation $\dfrac{x^2-1}{x-3} = 0$, which amounts to solving $x^2 - 1 = 0$; then we must exclude the values that make the denominator zero. The solutions to $x^2 - 1 = 0$ are $x_1 = -1$ and $x_2 = 1$, both valid.

   To solve $\dfrac{x^2-1}{x-3} > 0$, we need to understand the sign of $f_1(x) = x^2 - 1$ and of $f_2(x) = x - 3$; these maps have the same sign on $(3, +\infty)$ (positive) and on $(-1, 1)$ (negative).

   Therefore we can say that the initial inequality holds on $[-1, 1] \cup (3, +\infty)$.

2. We solve $\dfrac{1-x^2}{x^2-2x-1} < 0$.

   Let's consider the signs of $g_1(x) = 1 - x^2$ and $g_2(x) = x^2 - 2x - 1$. These maps have different sign on $(-\infty, -1)$, on $(1-\sqrt{2}, 1)$ and on $(1+\sqrt{2}, +\infty)$. The inequality's solution is therefore given by the union of these three intervals.

6 Equations and inequalities with absolute values

Given $|f(x)| = c$, there are three possibilities:

- $c < 0 \Rightarrow$ no solution;
- $c = 0 \Rightarrow$ the equation becomes $f(x) = 0$;
- $c > 0 \Rightarrow$ the solution set is the union of the solutions of the two systems:

$$\begin{cases} f(x) = c \\ f(x) \geq 0 \end{cases} \qquad \begin{cases} -f(x) = c \\ f(x) < 0 \end{cases}$$

As far as inequalities are concerned, consider $|f(x)| \leq c$. We have three cases:

- $c < 0 \Rightarrow$ no solution;
- $c = 0 \Rightarrow$ the solutions are those of the equation $f(x) = 0$;
- $c > 0 \Rightarrow$ the inequality is equivalent to $-c \leq f(x) \leq c$.

If, instead, we consider $|f(x)| \geq c$, these are the possibilities:

- $c \leq 0 \Rightarrow$ the inequality is true for any $x \in \mathbb{R}$;
- $c > 0 \Rightarrow$ the solution set is the union of the solution sets of

$$f(x) \geq c \qquad \text{and} \qquad f(x) \leq -c$$

6.1 Detailed examples

1. We solve $|x + 3| = 1$.

   By the definition of absolute value we obtain the equations $-x - 3 = 1$, if $x < -3$, and $x + 3 = 1$, if $x \geq -3$. Solving them, and bearing in mind the restrictions, we get the solutions $x_1 = -4$ and $x_2 = -2$.

2. Let's solve $|x^2 - 5x + 6| + x^2 = 0$.

   We rewrite the equation as $|x^2 - 5x + 6| = -x^2$. The left-hand side is positive or zero, whereas the right-hand side is negative or zero. They can be equal only if both vanish. But this cannot happen, because the zeroes of the left-hand side are $x = 2$, $x = 3$, while the term on the right is null only at the origin. Hence the starting equation hasn't got solutions.

7 Irrational equations and inequalities

Consider an equation of the type

$$\sqrt[n]{f(x)} = \sqrt[m]{g(x)}$$

where f, g are given maps and $n > 1,\ m \geq 1$.

It can be solved, assuming $x \in$ dom$f$ $\cap$ dom$g$, by raising to the right power. Let's see how through examples.

1. We wish to solve $\sqrt[3]{x^3 + 4} - 1 = x$.

   Since there's an odd root, the map's domain is the whole $\mathbb{R}$. First, we must isolate the root to one side, $\sqrt[3]{x^3 + 4} = x + 1$, and then raise to the power 3, $\left(\sqrt[3]{x^3 + 4}\right)^3 = (x + 1)^3$. A few computations yield the solutions $x_1 = \frac{-1+\sqrt{5}}{2}$ and $x_2 = \frac{-1-\sqrt{5}}{2}$.

2. Solve $\sqrt{2x - 1} = x - 2$

   Let's impose that the radicand is non-negative, so $x \geq \frac{1}{2}$. Notice that the left-hand side is certainly non-negative, hence also the right side must be so. This further implies $x \geq 2$. Squaring both sides and solving the quadratic equation produces $x_1 = 1$ and $x_2 = 5$, but only the latter is valid because of the constraints.

3. Solve $\sqrt{x - 1} + \sqrt{x + 1} = \sqrt{6 - x}$.

   The domain is the intersection of the domains of the three roots, that is the interval $[1, 6]$. Squaring leads to $2\sqrt{(x - 1)(x + 1)} = 6 - 3x$. As the right-hand side must be non-negative, we have $x \leq 2$. Keeping the previous constraint in mind, x must belong to [1,2]. Now we square once more and get $4x^2 - 4 = 36 - 36x + 9x^2$, whose roots are $x_1 = \frac{18-2\sqrt{31}}{5}$ and $x_2 = \frac{18+2\sqrt{31}}{5}$. Only x_1 is contained in [1, 2] and thus acceptable.

The study of irrational inequalities must be carried out with extreme care; let's look at

$$\sqrt[n]{f(x)} > \sqrt[m]{g(x)}$$

where f, g are given maps and $n > 1,\ m \geq 1$.

- Odd roots

  There are no problems with the domain; we just raise everything to the suitable power. For instance, $\sqrt[3]{f(x)} > g(x)$ becomes $f(x) > g(x)^3$.

- Even roots

  Here we must mind the functions' domains and the "hidden" constraints: for instance,

$$\sqrt{f(x)} < g(x)$$

is equivalent to the system

$$\begin{cases} f(x) \geq 0 \\ g(x) > 0 \\ f(x) < (g\,(x))^2 \end{cases}.$$

The inequality

$$\sqrt{f(x)} > g(x)$$

reduces to the two systems

$$\begin{cases} f(x) \geq 0 \\ g(x) < 0 \end{cases} \quad \begin{cases} g(x) \geq 0 \\ f(x) > (g\,(x))^2 \end{cases}.$$

The required solution is the union of the solutions of these systems.

Explicitly, let's solve $\sqrt{x-1} > 12 - 2x$.

We need to consider only $x \geq 1$. The inequality is clearly satisfied when the right-hand side is negative, hence when $x > 6$. If $x \leq 6$ we may square (both sides are non-negative) to get $x - 1 > 144 + 4x^2 - 48x$, hence $4x^2 - 49x + 145 < 0$. The solution to the latter is given by the interval $(5, 29/4)$; therefore, the inequality holds on $(5, +\infty)$.

8 EXERCISE - RATIONAL AND IRRATIONAL EQUATIONS AND INEQUALITIES

TRUE OR FALSE?

1. The equation $0x = 0$ hasn't got solutions. **T** **F**
2. The equation $0x = 1$ doesn't have solutions. **T** **F**
3. The solution to $5x - 3 = 0$ is $x = -2$. **T** **F**
4. The solution to $4x = 0$ is $x = \frac{1}{4}$. **T** **F**
5. The equation $x(x^2 + 1) = 0$ has solution $x = 0$. **T** **F**
6. $-5(x-1)(x+3)(x^2+10) = 0$ is equivalent to $(x+1)(x+3) = 0$. **T** **F**
7. The equation $\left(x^2+5\right)^2 \left(x^2+2\right)^2 = 0$ has no rational solutions. **T** **F**
8. $\frac{x-4}{2x-6} = 0$ is solved by $x = 4$ and $x = 3$. **T** **F**
9. $\frac{1}{x-2} = 3$ is equivalent to $x - 2 = \frac{1}{3}$. **T** **F**
10. The discriminant of $3x^2 - 5x + 4 = 0$ is negative. **T** **F**
11. $|x| = 2$ is equivalent to $x^2 = 4$. **T** **F**

EXERCISE 1

Let f be a real map of one real variable. Tell which of the following statements are equivalent to $f(x) = 0$:

1. $f(x) + x = x$
2. $f(x) + \frac{1}{x^2-3} = \frac{1}{x^2-3}$
3. $(x^2-1)f(x) = 0$
4. $(x^2+1)f(x) = 0$
5. $f^2(x) = 0$
6. $f(x) - x^2 = x^4$
7. $\frac{f(x)}{x^4+1} = 0$
8. $\frac{f(x)}{x-3} = 0$

Exercise 2

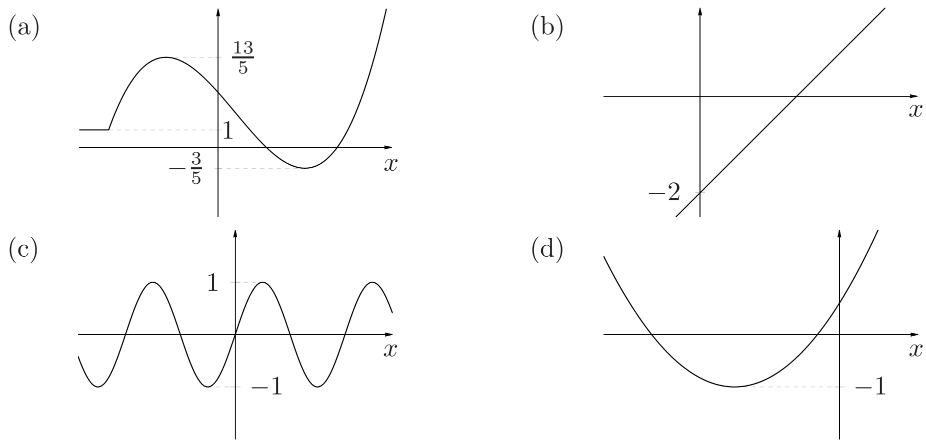
Given real maps f and g defined on $\mathbb{R}$, say which inequalities are equivalent to $f(x) < g(x)$:

1. $f(x) + g(x) > 0$
2. $f(x)g(x) < 0$
3. $2 - g(x) < 2 - f(x)$
4. $(1 + x^2)f(x) < (1 + x^2)g(x)$
5. $\left(x^2 - 1\right) g(x) > \left(x^2 - 1\right) f(x)$
6. $\dfrac{f(x)}{x+3} < \dfrac{g(x)}{x+3}$
7. $f(x)g(x) < (g(x))^2$
8. $(f(x))^2 < (g(x))^2$

Exercise 3

Consider the four graphs of the map f below, one at a time. Tell if the equation $f(x) = k$ satisfies the following properties (if any):

1. there's no solution, whichever $k \in \mathbb{R}$;
2. for some value $k \in \mathbb{R}$ there's no solution;
3. for every $k \in \mathbb{R}$ there's exactly one solution;
4. for some $k \in \mathbb{R}$ there are at least two solutions;
5. there isn't any solution when $k = -3$, and three solutions when $k = 2$;
6. for all $k \in \mathbb{R}$ there are two solutions.



Exercise 4

Solve and interpret geometrically the following equations:

1. $-3x + 7 = 2x + \frac{3}{4}$
2. $\frac{3}{2}x - 5 = 2(1 - x)$
3. $\frac{2}{3}x - 1 = \left(-\frac{3}{2}\right) x + 25$
4. $2x + 3 = 2x - 5$
5. $\frac{x+2}{1+\frac{1}{3}} = \frac{x-2}{1-\frac{1}{3}}$
6. $x^2 - 2 = |x|$

Exercise 5

Solve and interpret geometrically the following inequalities:

1. $x + \frac{1}{3} < -\frac{2}{3}x + \frac{1}{2}$
2. $\frac{1}{5}x + \frac{1}{2} < \frac{2x-1}{5-\frac{1}{2}}$
3. $3x + \frac{1}{3} < 3x + 2$
4. $-x - 3 > \frac{-\frac{5}{4}x+3}{\frac{3}{4}+\frac{1}{2}}$

Exercise 6

Solve on $\mathbb{R}$ the following equations of degree two, providing a geometric explanation based *only* on the graphs of the maps appearing on either side:

1. $x^2 - 2x + 3 = 2x$
2. $x^2 - 8x + \frac{1}{2} = -x^2 + 8x - \frac{1}{2}$
3. $x^2 + 4x - \frac{2}{3} = x^2 - 3x + 1$
4. $2x^2 - 4x + 3 = -3x^2 + 12x - 13$

Exercise 7

Choose values for the coefficients a, b, c of $f(x) = ax^2 + bx + c$ so that the set where f is positive is:

1. $\mathbb{R}$
2. (-2,3)
3. $\emptyset$;
4. $(-\infty, 1) \cup (5, \infty)$
5. $(-\infty, 2) \cup (2, \infty)$

Exercise 8

Solve the following inequalities:

1. $\dfrac{x^2+5x+4}{x^4+1} > 0$
2. $\dfrac{x^3+8}{x^2-1} > 0$
3. $\dfrac{x-2}{|2x+1|} > -\frac{1}{3}x$
4. $\dfrac{|x-1|}{|3x+1|} \leq 1$

Exercise 9

Solve the following inequalities and interpret them geometrically:

1. $|x+2| = 1$
2. $|x+5| = -1$
3. $|x-1| + |2x+1| = 10$
4. $\left|x^2-1\right| - \left|x^2-5\right| = 3$

Exercise 10

Tell under which conditions the roots are well defined, and transform them into roots of the same index:

1. $\sqrt{a},\ \sqrt[12]{a^5},\ \sqrt[4]{a^3}$
2. $\sqrt[3]{x-y},\ \sqrt[5]{x+y},\ \sqrt[4]{x^2-y^2}$

Exercise 11

Rationalize[1] the following expressions:

1. $\dfrac{5}{\sqrt[3]{54}}$
2. $\dfrac{1}{3-\sqrt{2}}$
3. $\dfrac{1-\sqrt{\pi+1}}{1+\sqrt{\pi+1}}$
4. $\dfrac{3\sqrt{2}}{2\sqrt{3}-3\sqrt{2}}$
5. $\dfrac{2}{\sqrt[3]{5}-\sqrt[3]{3}}$

Exercise 12

Solve the following irrational inequalities and equations:

1. $2\sqrt{x-1} - x = 0$
2. $\sqrt{x+3} = 1-3x$
3. $\sqrt{2x+6} - x + 1 = 0$
4. $3\sqrt{x+2} - x - 4 = 0$
5. $\sqrt[3]{x+4} = 3$
6. $\sqrt{x-1} - \sqrt{2x-3} = 0$
7. $\sqrt{5x-6} > x$
8. $\sqrt{x+2} + \sqrt{3x-1} > 0$
9. $\sqrt{\frac{x-4}{x+2}} < 2$

Exercise 13

Discuss using graphs:

[1] *Rationalizing* means getting rid of a root appearing in a denominator by multiplying and dividing the ratio simultaneously by a suitable factor. For instance, $3/\sqrt{2}$ can be rationalized if we multiply and divide by $\sqrt{2}$, while $3/(\sqrt{7}-\sqrt{2})$ gets rationalized by using the factor $\sqrt{7}+\sqrt{2}$.

1. $\sqrt{x+1} \geq x-3$
2. $\sqrt{x-2} > -1$
3. $\sqrt{-x-1} > 0$
4. $\sqrt{x+5} \geq 2$

EXERCISE 14

Determine domain and positivity set for:

1. $f(x) = \sqrt{x-2} + 1$
2. $f(x) = \sqrt{x+3} + \sqrt{x^2+9}$
3. $f(x) = \sqrt[3]{x^2-1}$
4. $f(x) = \sqrt[3]{x-1} + \sqrt[3]{x-2}$
5. $f(x) = \frac{\sqrt{x}-1}{\sqrt{|x|-2}}$
6. $f(x) = \sqrt[3]{x+2} - \sqrt{x+2}$
7. $f(x) = \frac{\sqrt{x-5}+\sqrt{2x+1}}{\sqrt[3]{1-x}}$
8. $f(x) = \sqrt[4]{x^4-1} - x^2$
9. $f(x) = \frac{\sqrt{x-1}\sqrt{x+2}}{\sqrt{6x^2+x-2}}$

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9 SOLUTIONS

TRUE OR FALSE?

True: 2, 5, 7, 9, 10, 11

EXERCISE 1

Equivalent: 1, 4, 5, 7

EXERCISE 2

Equivalent: 3, 4

EXERCISE 3

1. none
2. (a), (c), (d)
3. (b)
4. (a), (c), (d)
5. (a)
6. none

EXERCISE 4

1. $\frac{5}{4}$
2. 2
3. 12
4. $\emptyset$
5. 6
6. $\{-2, 2\}$

EXERCISE 5

1. $(-\infty, \frac{1}{10})$
2. $(\frac{65}{22}, +\infty)$
3. $\mathbb{R}$
4. $\emptyset$

EXERCISE 7

1. $a > 0$, $\Delta < 0$ (eg $a = 1$, $b = 0$, $c = 1$)
2. $a = -1$, $b = 1$, $c = 6$ (more generally, $a = -y$, $b = y$, $c = 6y$ with $y > 0$)
3. $a < 0$, $\Delta \leq 0$ (for example $a = -1$, $b = 0$, $c = -1$)
4. $a = 1$, $b = -6$, $c = 5$ (more generally, $a = y$, $b = -6y$, $c = 5y$ with $y > 0$)
5. $a = 1$, $b = -4$, $c = 4$ (more generally, $a = y$, $b = -4y$, $c = 4y$ with $y > 0$)

EXERCISE 8

1. $(-\infty, -4) \cup (-1, +\infty)$
2. $(-2, -1) \cup (1, +\infty)$
3. $(1, +\infty)$
4. $(-\infty, -1] \cup [0, +\infty)$

EXERCISE 9

1. $\{-3, -1\}$
2. $\emptyset$
3. $\{-10/3, 10/3\}$
4. $\{-3\frac{\sqrt{2}}{2}, 3\frac{\sqrt{2}}{2}\}$

EXERCISE 10

1. All defined for $a \geq 0$, $\sqrt[12]{a^6}$. $\sqrt[12]{a^5}$, $\sqrt[12]{a^9}$
2. Defined for $\forall x, y \in \mathbb{R}$, $\forall x, y \in \mathbb{R}$, $\{-|x| \leq y \leq |x|\}$. $\sqrt[60]{(x-y)^{20}}$, $\sqrt[60]{(x+y)^{12}}$, $\sqrt[60]{(x^2-y^2)^{15}}$ respectively.

Exercise 11

1. $\frac{5\sqrt[3]{54^2}}{54}$
2. $\frac{3+\sqrt{2}}{7}$
3. $\frac{2\sqrt{\pi+1}-\pi-2}{\pi}$
4. $-3-\sqrt{6}$
5. $\sqrt[3]{25}+\sqrt[3]{15}+\sqrt[3]{9}$

Exercise 12

1. 2
2. $-2/9$
3. 5
4. $-1, 2$
5. 23
6. 2
7. $(2, 3)$
8. $[\frac{1}{3}, +\infty)$
9. $[-\infty, -4) \cup [4, +\infty)$

Exercise 14

The domains and positivity sets are, respectively:

1. $[2, +\infty)$, $[2, +\infty)$
2. $[-3, +\infty)$, $[-3, +\infty)$
3. $\mathbb{R}$, $(-\infty, -1) \cup (1, +\infty)$
4. $\mathbb{R}$, $(\frac{3}{2}, +\infty)$
5. $[0, 4) \cup (4, +\infty)$, $[0, 1) \cup (4, +\infty)$
6. $[-2, +\infty)$, $[-2, -1)$
7. $[5, +\infty)$, $\emptyset$
8. $(-\infty, -1] \cup [1, +\infty)$, $\emptyset$
9. $[1, +\infty)$, $(1, +\infty)$