Politecnico di Torino

Bridging course in mathematics

Sheet 2

Elementary functions

1 Functions

1.1 Definitions

A *real(-valued) function of one real variable* is a rule f that assigns to a real number x at most one real number y; to indicate this one writes $y = f(x)$. Functions are also called *maps*.

Example 1.1 *The rule "associate to each real number its triple plus one" defines the function* $y = 3x + 1$*; for every real* x *we can determine in a unique way the corresponding value of* y*.*

Example 1.2 *The law "each real number is mapped to its square root" is uniquely defined; in contrast to above, though, it cannot be applied to each real number, but only to non-negative ones.*

Example 1.3 *The assignment "to each number* x *associate the real numbers whose square is* x*" doesn't define a function; for instance,* $x = 9$ *would be mapped to* 3 *but also to* -3*; this law is therefore not uniquely defined.*

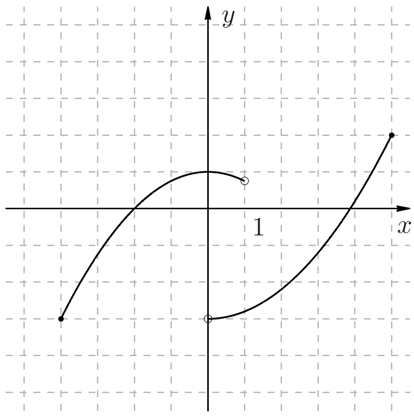
Let's fix an $x_0 \in \mathbb{R}$ and the corresponding $y_0 = f(x_0)$ obtained using the map f, and let's consider the ordered pair $(x_0, f(x_0))$ in the Cartesian product $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$. If we do this for each $x \in \mathbb{R}$ we are identifying a subset of $\mathbb{R}^2$ called the *graph* of the map f.

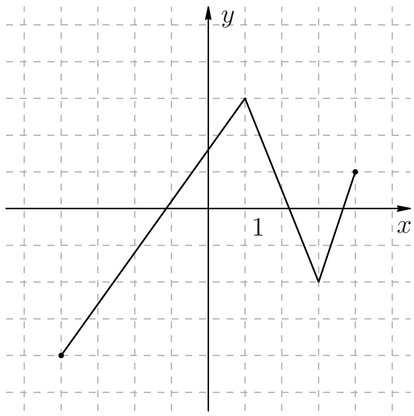
The projection of the graph onto the x-axis is the *domain* of the function, whereas the projection onto the y-axis is the *range* of the function.

Example 1.4 *Consider the subsets of* $\mathbb{R}^2$ *shown in the picture (by convention, a thick dot means that the point belongs to the graph, while a hollow dot means the point is excluded).*

The left picture is not the graph of a function. In fact, to each $x \in (0, 1)$ *correspond two values of* y *(one negative, one positive).*

The right one, instead, satisfies the uniqueness property; projecting the graph onto the axes gives the function's domain $[-4, 4]$*, and range* $[-4, 3]$*.*





1.2 Examples of maps

1. **Constant maps**

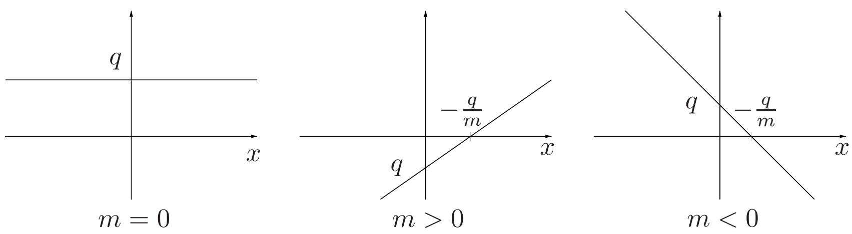
   A function f is said *constant* when it assumes the same value on all elements of its domain.

   If $\mathrm{dom}(f) = \mathbb{R}$, the graph of the constant function $y = c$ is the straight line parallel to the x-axis, which is made of all the points with second coordinate equal to c.

2. **Linear and affine maps**

   A *linear* map is described by direct-proportionality relationship $y = mx$, which has a straight line through the origin as graph; the number m (proportionality factor) is the *slope* of the line.

   An *affine* map has equation $y = mx + q$. The graph of such a function is a line passing through the point $(0, q)$ with slope m.

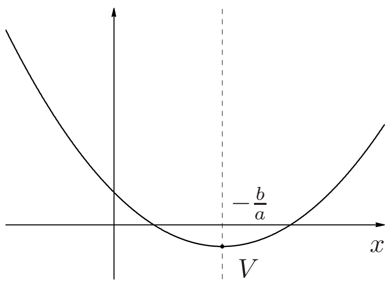


3. **Quadratic maps**

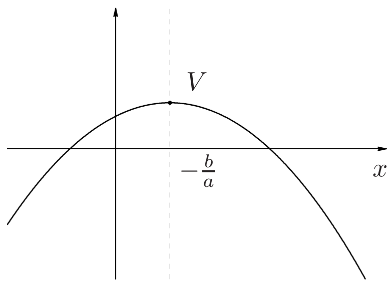
   The simples quadratic map is $y = x^2$. Its graph is a parabola with vertex at the origin, symmetry axis coinciding with the y-axis, and convex (intuitively, "U-shaped"); the map's domain is $\mathbb{R}$ and its range $[0, +\infty)$.

   In general, every map of type $y = ax^2 + bx + c$ (with $a \neq 0$) is *quadratic*, and has a *parabola* as graph, with the following features

- the vertex V has coordinates $\left(-\frac{b}{2a}, \frac{4ac-b^2}{4a}\right)$;
- the symmetry axis is the line $x = -\frac{b}{2a}$;
- the parabola is convex if $a > 0$, concave if $a < 0$;



$a > 0$

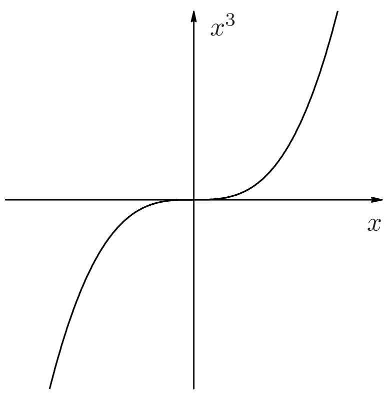


$a < 0$

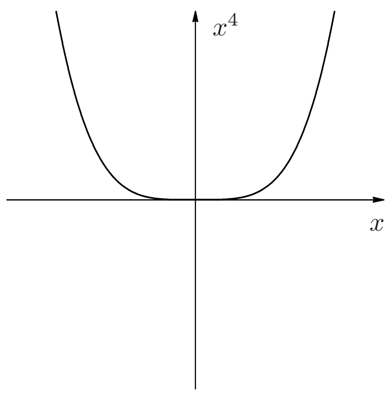
4. **Power functions** $y = x^n$

   We have seen that for $n = 1$ the graph is a straight line, and for $n = 2$ we have a parabola. For $n = 0$ the map is constant and equals 1 for all $x < 0$ and $x > 0$; at the point $x = 0$ it is not defined.

   Apart from these cases, the functions in the family $y = x^n$, with $n \in \mathbb{N}$, called *powers*, have different graphs according to whether n is even or odd.

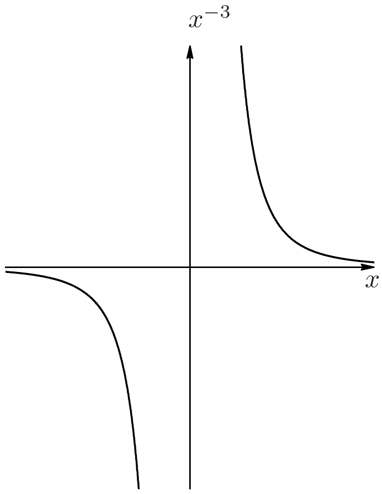


n positive and odd

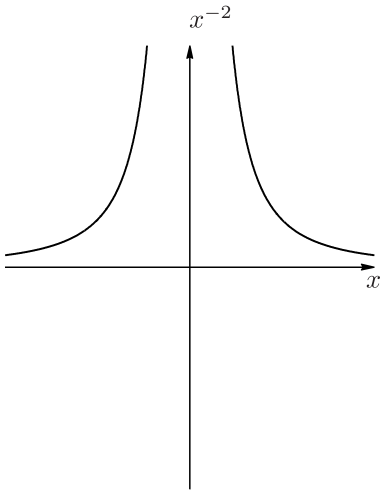


n positive and even

   If n is negative and odd, the graph is similar to the one of the hyperbola (below, left), whereas if n is negative and even it goes as in the right figure below.



n negative and odd



n negative and even

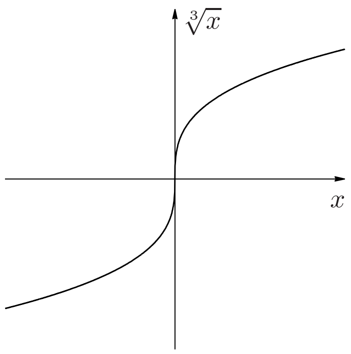
Particularly important is the case $n = -1$; this is the inverse-proportionality law, which, in general, reads $y = \frac{k}{x}$, with $k \neq 0$. The map defined in this way has domain $\mathbb{R} \setminus \{0\}$. Its graph is the hyperbola whose asymptotes are the coordinate axes.

5. **Root functions**

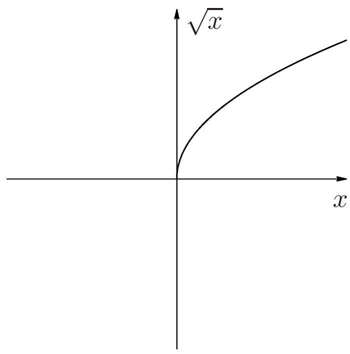
   The behaviour of root functions $y = \sqrt[n]{x}$ depends upon the parity of n.

   If n is odd and $n \geq 3$ then the map $y = \sqrt[n]{x}$ has domain and range equal $\mathbb{R}$ (the left figure shows the case $n = 3$).

   If n is even one can define the function $\sqrt[n]{x}$ only for $x \in [0, \infty)$, and the range is $[0, \infty)$ (the case $n = 2$ is shown on the right).



n odd

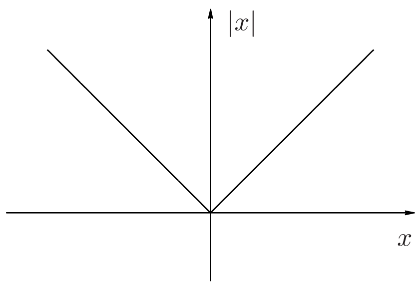


n even

6. **Absolute value function**

   The absolute value function is defined as

$$|x| = \begin{cases} -x & \text{for } x < 0 \\ x & \text{for } x \geq 0 \end{cases}$$



2 Plane transformations and graphs

Starting with the graphs already seen we can draw other maps using simple operations called *transformations of the plane*.

We will not discuss plane transformations in full, but rather see how to sort out the most common situations.

1. Vertical translation (by q)

$$y = f(x) \Longrightarrow y = f(x) + q$$

2. Horizontal translation (by p)

$$y = f(x) \Longrightarrow y = f(x - p)$$

3. Symmetry with respect to the x-axis

$$y = f(x) \Longrightarrow y = -f(x)$$

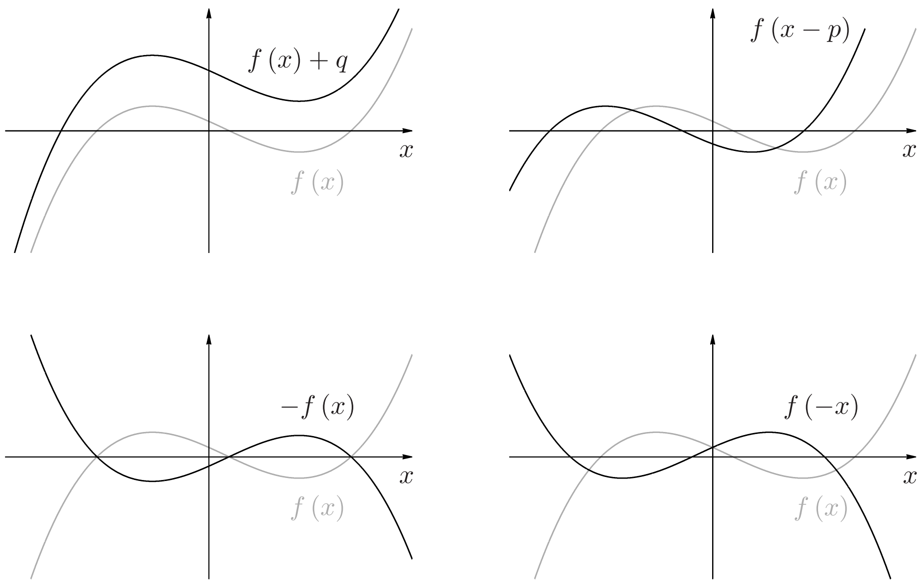
4. Symmetry with respect to the y-axis

$$y = f(x) \Longrightarrow y = f(-x)$$

This last symmetry suggests the following definition.

Definition 2.1 *Let f be a map with symmetric domain with respect to the origin (meaning that $x \in \mathrm{dom}\ f$ implies $-x \in \mathrm{dom}\ f$);*

- *if $f(-x) = f(x)$ for every $x \in \mathrm{dom}\ f$, the map is called* even*;*
- *if $f(-x) = -f(x)$ for every $x \in \mathrm{dom}\ f$, the map is called* odd.



Example 2.2 *Consider the parabola of equation $y = x^2$ and let's shift it horizontally by $p = -1$; this means that the graph moves leftwards, and the equation becomes $y = (x + 1)^2$.*

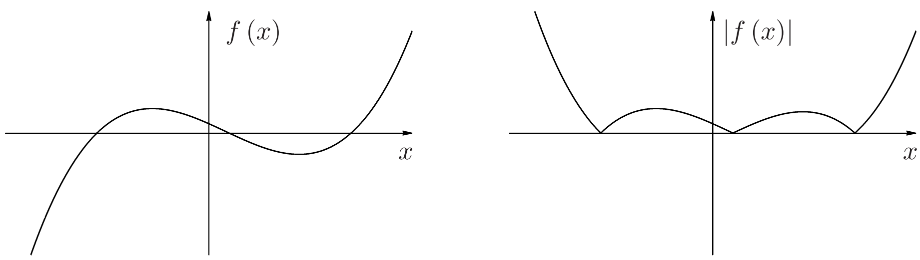
Now let's shift vertically with $q = 2$ (upwards); the new equation reads $y = (x + 1)^2 + 2 = x^2 + 2x + 3$.

3 The graphs of $y = |f(x)|$ and $y = f(|x|)$

Given the graph of $y = f(x)$ we obtain the graph of $y = |f(x)|$ using the definition of absolute value:

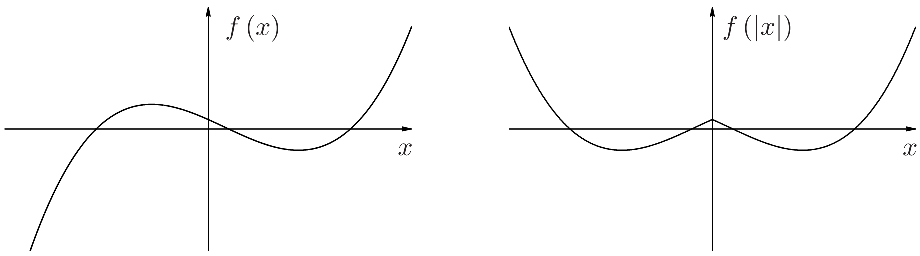
$$|f(x)| = \begin{cases} f(x) & \text{if } f(x) \geq 0 \\ -f(x) & \text{if } f(x) < 0 \end{cases} .$$

On the left we see the graph of $y = f(x)$, on the right the graph of $y = |f(x)|$.



The part of graph on the half-plane where y is negative gets reflected on the positive half-plane.

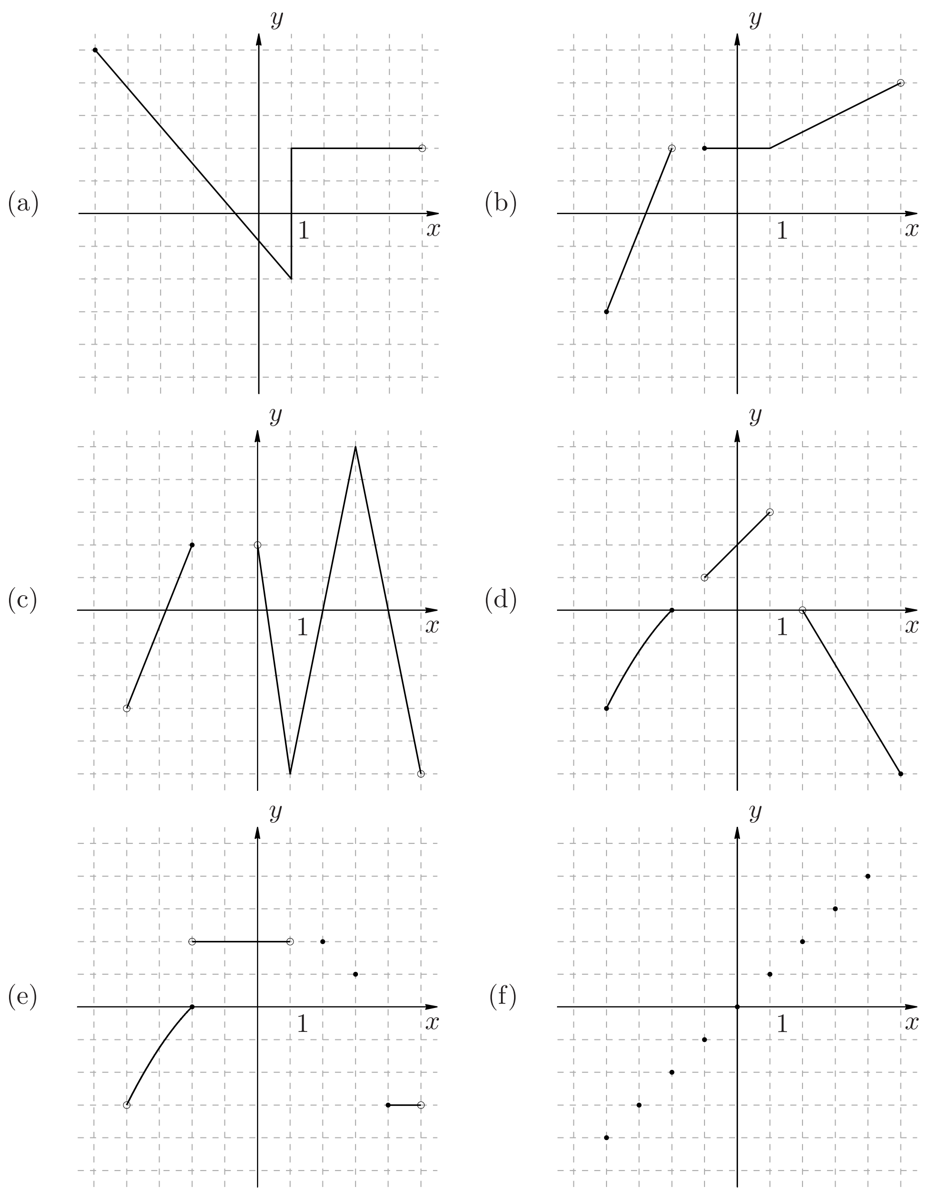
Through similar arguments we can draw the graph of $f(|x|)$. Let's observe, to begin with, that in order to define $f(|x|)$ it is necessary that the function's domain contains points with positive or null first coordinate. To build the graph of $y = f(|x|)$ from that of $y = f(x)$ we reproduce the graph of $y = f(x)$ when $x \geq 0$, while for negative values of x we draw the symmetric curve with respect to the y-axis. Let's look at an example:



4 EXERCISES – ELEMENTARY FUNCTIONS

Exercise 1

Find, among the subsets of the plane below, which are graphs of functions. Of the latter determine domain and range.



Exercise 2

Find the translations that map the vertex of the parabola $y = x^2$ to the points $V' = (-5, 3)$ and $V'' = (3, -3)$.

Exercise 3

Given the map $y = |x|$, translate it with $p = -1$ and $q = 3$, and with $p = 2/3$ and $q = -3/2$. Draw the graphs of the two maps obtained in this way.

Exercise 4

Determine the translation that moves $y = |x|$ to $y = |x-2| + 3$ and to $y = \left|x - \frac{1}{2}\right|$.

Exercise 5

Given the functions $y_1(x) = |x|$, $y_2(x) = x^2 - x$, $y_3(x) = x^2 - 2x + 3$ find the explicit expression and draw the graph of the functions symmetric to these with respect to the axis x, and then do the same with respect to the axis y.

Exercise 6

Find a quadratic map with maximum equal 3, having maximum point at -2 and that vanishes at $x = 0$.

Exercise 7

Draw the graph of the function $y = -|-x+3| + 2$ indicating the transformations that lead to it starting from the graph of $y = |x|$.

Exercise 8

Draw the graphs of the maps

1. $y = (x+2)^3$
2. $y = \dfrac{1}{(x-1)^2} + 3$
3. $y = \dfrac{1}{(-x+2)^5}$

indicating the transformations that produce them if we start from the graphs of $y = x^3$, $y = \frac{1}{x^2}$, and $y = \frac{1}{x^5}$ respectively.

Exercise 9

Sketch the graphs of

$$f(x) = 2x - 1, \qquad g(x) = x^2 - 5x + 4, \qquad h(x) = \frac{-2x+3}{x-5}$$

and use them to determine the graphs of:

1. $f_1(x) = |2x - 1|$
2. $f_2(x) = 2|x| - 1$
3. $f_3(x) = |2|x| - 1|$

4. $g_1(x) = \left|x^2 - 5x + 4\right|$
5. $g_2(x) = x^2 - 5|x| + 4$
6. $g_3(x) = \left|x^2 - 5|x| + 4\right|$
7. $h_1(x) = \left|\dfrac{-2x+3}{x-5}\right|$

EXERCISE 10

Starting with the graphs of $y = \sqrt{x}$ and $y = \sqrt[3]{x}$ and using suitable plane transformations, construct the graphs of the following maps:

1. $f(x) = \sqrt{x} + 2$
2. $f(x) = \sqrt{x+2}$
3. $f(x) = \sqrt[3]{x} + 2$
4. $f(x) = \sqrt[3]{x+2}$
5. $f(x) = \sqrt{3-x}$
6. $f(x) = 3 - \sqrt[3]{x}$

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5 SOLUTIONS

Exercise 1

(a) isn't the graph of a map, while all the others are.
(b) dom $f = [-4, -2) \cup [-1, 5)$; im $f = [-3, 4)$;
(c) dom $f = (-4, -2] \cup (0, 5)$; im $f = [-5, 5]$;
(d) dom $f = [-4, -2] \cup (-1, 1) \cup (2, 5]$; im $f = [-5, 0] \cup (1, 3)$;
(e) dom $f = (-4, 1] \cup \{2, 3\} \cup [4, 5)$; im $f = [-3, 0] \cup \{1, 2\}$;
(f) dom f =im $f = \{-4, -3, -2, -1, 0, 1, 2, 3, 4\}$.

Exercise 2

Using the notation of section 2: vertex at $V' \Longrightarrow p = -5, q = 3$, vertex at $V'' \Longrightarrow p = 3, q = -3$

Exercise 4

With the notation of section 2: function $y = |x - 2| + 3 \Longrightarrow p = 2, q = 3$, function $y = |x - 1/2| \Longrightarrow p = 1/2, q = 0$.

Exercise 5

The symmetric maps with respect to the x-axis are $y_1(x) = -|x|$, $y_2(x) = x - x^2$ and $y_3(x) = -x^2 + 2x - 3$ respectively; the symmetric functions with respect to the y-axis are $y_1(x) = |x|$, $y_2(x) = x^2 + x$ and $y_3(x) = x^2 + 2x + 3$.

Exercise 6

A quadratic map with maximum 3 and maximum point at -2 is of the form $y(x) = \alpha(x + 2)^2 + 3$; imposing the requirement $y(0) = 0$ we obtain $\alpha = -3/4$.
