Politecnico di Torino

Bridging course in mathematics

Sheet 1

Polynomials

1 Start-up: sets of numbers

The following symbols will be used in the sequel to indicate numerical sets:

- *natural* numbers, denoted by $\mathbb{N} = \{0, 1, 2, 3, ...\}$;
- *integer* numbers, denoted by $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, +1, +2, +3, ...\}$;
- *rational* numbers, denoted by $\mathbb{Q}$; a rational number can be written as a ratio m/n of two integers, with $n \neq 0$.
- *real* numbers, denoted by $\mathbb{R}$; the set of real numbers contains the rational numbers as well as the irrational ones.

We shall usually work with real numbers.

2 Definitions

A *polynomial* in the variable x with real coefficients (a real polynomial in x, for short) is an expression of the form

$$A_n(x) = a_0 + a_1 x + ... + a_n x^n,$$

where $a_0, a_1, ..., a_n$ are real numbers (called the *coefficients* of the polynomial) and $a_n \neq 0$ (the *leading* coefficient). The summands are called *monomials*. The *degree* of a polynomial is the maximum exponent of the non-zero monomials appearing in the expression. Above, having assumed $a_n \neq 0$, the degree equals n.

A real number c such that $A_n(c) = 0$ is called a *root*, or *zero*, of the polynomial.

Two polynomials are equal if they have the same degree and the coefficients of the monomials of equal degree coincide.

3 Operations between polynomials

3.1 Sum and product

The *sum* of two polynomials is the polynomial obtained by adding the coefficients of the monomials with the same degree. For example:

$$(x^2+2x-5)+(x^3-x+2)=(0x^3+x^2+2x-5)+(x^3+0x^2-x+2)=$$
$$(0+1)x^3+(1+0)x^2+(2-1)x+(-5+2)=x^3+x^2+x-3.$$

In a similar way one defines the difference of two polynomials.

The *product* is the polynomial of the form $A_n(x)\cdot B_m(x)=C_{n+m}(x)=c_0+c_1x+...+c_{n+m}x^{n+m}$, where the coefficients are given by:

$$c_0=a_0b_0,\quad c_1=a_1b_0+a_0b_1,\quad c_k=a_0b_k+a_1b_{k-1}+a_2b_{k-2}+....a_{k-1}b_1+a_kb_0.$$

Let's see how to compute the product in practice:
$(x-1)(x^2+x+1)=x^2(x-1)+x(x-1)+(x-1)=x^3-x^2+x^2-x+x-1=x^3-1.$

Detailed exercises

1. $(3x^5+x^3)+(x-1+x^4-5x^3)=$
   $3x^5+x^4+x^3-5x^3+x-1=3x^5+x^4-4x^3+x-1$

2. $(x^3+1)(3x^2+x-5)=$
   $x^3\cdot 3x^2+x^3\cdot x+x^3\cdot(-5)+1\cdot 3x^2+1\cdot x+1\cdot(-5)=3x^5+x^4-5x^3+3x^2+x-5$

3. $(x^2-1)(x^2+1)=x^4-x^2+x^2-1=x^4-1$

4. Simplify the expression $\left(\frac{x^2-y^2}{x+y}+y\right)\frac{y}{x-y}\left(\frac{1}{x}+\frac{1}{y}\right)$:
   $\left(\frac{(x+y)(x-y)}{x+y}+y\right)\frac{y}{x-y}\frac{x+y}{xy}=(x-y+y)\frac{y}{x-y}\frac{x+y}{xy}=\frac{x+y}{x-y}$

5. Determine A, B, C so that $\frac{3x^2+6}{x^3-x^2-6x}=\frac{A}{x}+\frac{B}{x-3}+\frac{C}{x+2}$:

$$\begin{aligned}\frac{A}{x}+\frac{B}{x-3}+\frac{C}{x+2}&=\frac{A(x-3)(x+2)+Bx(x+2)+Cx(x-3)}{x^3-x^2-6x}\\&=\frac{(A+B+C)x^2+(-A+2B-3C)x-6A}{x^3-x^2-6x}\end{aligned}$$

$$\begin{cases}(A+B+C)x^2=3x^2\\(-A+2B-3C)x=0\cdot x\\-6A=6\end{cases}$$

$$\begin{cases}A+B+C=3\\-A+2B-3C=0\\-6A=6\end{cases}$$

Answer: $A=-1$, $B=11/5$, $C=9/5$

3.2 Dividing polynomials

Given two polynomials $A_n(x), B_m(x)$ of degree n and m respectively, with $n \geq m$, there exist polynomials $Q(x)$ and $R(x)$, called the *quotient* and *remainder*, such that:

- the degree of $R(x)$ is less than m;
- we have
$$A_n(x) = B_m(x)Q(x) + R(x). \tag{1}$$
  This can be written as follows
$$\frac{A_n(x)}{B_m(x)} = Q(x) + \frac{R(x)}{B_m(x)}. \tag{2}$$

Definition 3.1 *If the remainder $R(x)$ in equation* (1) *is the zero polynomial, one says that $B_m(x)$ divides $A_n(x)$, or that $A_n(x)$ is divisible by $B_m(x)$.*

In order to compute the quotient and the remainder we divide by decreasing degree; we shall describe this method by an example. Take $A(x) = 2x^4 + x^3 - x + 2$ of $\deg = 4$ and $B(x) = x^2 + 3$ of $\deg = 2$. This means the quotient will have degree $\deg Q = 4 - 2 = 2$ and the remainder $\deg R < 2$.

1. After ordering the monomials by decreasing degree, compute the quotient between the leading monomials of $A(x)$ and $B(x)$, and obtain $2x^2$.
2. Compute the product $2x^2 \cdot B(x)$ and subtract it from $A(x)$; this gives a polynomial of degree 3, say $R_3(x)$.
3. What we have done so far can be arranged in a grid:

$2x^4$	$+x^3$	$+0x^2$	$-x$	$+2$	x^2	$+3$
$2x^4$		$+6x^2$			$2x^2$	
$R_3(x) \rightsquigarrow$	$+x^3$	$-6x^2$	$-x$	$+2$		

4. Now repeat the procedure, and divide the leading monomial of $R_3(x)$ by x^2; the result x is added to $2x^2$. Continuing in the same way one finds the polynomial of degree two $R_2(x) = R_3(x) - xB(x)$.
5. Next, divide $R_2(x)$ by x^2, and get -6. Then compute $R_1(x) = R_2(x) - (-6)B(x)$.

   The grid now looks as follows.

$2x^4$	$+x^3$	$+0x^2$	$-x$	$+2$	x^2		$+3$
$2x^4$		$+6x^2$			$2x^2$	$+x$	-6
	$+x^3$	$-6x^2$	$-x$	$+2$			
$xB(x) \rightsquigarrow$	$+x^3$		$+3x$				
$R_2(x) \rightsquigarrow$		$-6x^2$	$-4x$	$+2$			
		$-6x^2$		-18			
$R_1(x) \rightsquigarrow$			$-4x$	$+20$			

The polynomial $R_1(x) = -4x + 20$ has degree smaller than the degree of $B(x)$, and is therefore the remainder of the division. Equation (1) in the present case reads:

$$2x^4 + x^3 - x + 2 = (2x^2 + x - 6)(x^2 + 3) + (-4x + 20).$$

Detailed exercises

1. $\dfrac{x^3 + 3x^2 - 13x - 15}{x + 5}$

x^3	$+3x^2$	$-13x$	-15		x	$+5$
x^3	$+5x^2$			x^2	$-2x$	-3
	$-2x^2$	$-13x$	-15			
	$-2x^2$	$-10x$				
		$-3x$	-15			
		$-3x$	-15			
			0			

Answer: $(x - 3)(x + 1)$

2. $\dfrac{x^6 + 15}{x^2 - 1}$

x^6			$+15$		x^2	-1
x^6	$-x^4$			x^4	$+x^2$	$+1$
	x^4		$+15$			
	x^4	$-x^2$				
		x^2	$+15$			
		x^2	-1			
			$+16$			

Answer: $(x^4 + x^2 + 1) + \dfrac{16}{x^2 - 1}$

4 Factorization

Factorizing a polynomial consists in writing it as a product of irreducible polynomials.

If we consider polynomials with real coefficients (as in Calculus), polynomials of degree one, and polynomials of degree two having negative discriminant are irreducible. For higher degree, it is more complicated to tell.

But remember that a polynomial cannot always be factorized in the practice.

We remind a few ways to decompose a polynomial.

- Finding a common factor: $x^4 - 3x^3 + 5x^2 = x^2(x^2 - 3x + 5)$.
- Using partial factors

  $$q(x) = x^4+a^2x^2+b^2x^2+a^2b^2 = x^2(x^2+a^2)+b^2(x^2+a^2) = (x^2+a^2)(x^2+b^2).$$

The presence of factors of degree one in a polynomial is easy to check, due to the following theorem.

Theorem 4.1 *$(x - c)$ divides the polynomial $A_n(x)$ if and only if $A_n(c) = 0$.*

Using this result we have:

a) the binomial $x^n - a^n$ can always be divided by $x - a$;
   if n is even also $x + a$ divides it.

b) The binomial $x^n + a^n$ is divisible by $x + a$ if n is odd;
   if n is even neither $x + a$, nor $x - a$, divide it.

Remark 4.2 *For polynomials with integer coefficients there are theorems that can help to find integer or rational roots:*

- *the integer roots (if any) should be looked for among the factors with sign of the constant term, including 1;*
- *the rational roots (if any) should be looked for among the rationals of the form $\pm p/q$, where p is a factor of the constant term, and q is a factor of the leading coefficient.*

4.1 Ruffini's method

To divide a polynomial by $x - c$ we use an algorithm, called *Ruffini-Horner algorithm*, which is conventionally written in a table.

As example, we will divide the degree-three polynomial $A(x) = 4x^3 - 5x + 7$ by $B(x) = x - 2$.

1. First of all, draw three lines as in the picture below: two vertical ones and one horizontal.
2. Begin by writing the coefficients of the monomials of $A(x)$ on the first row, ordered from the highest degree to the smallest; the constant term 7 should be written to the right of the second vertical bar.
3. On the second row write 2 (the constant term of $B(x)$) to the far left.

	4	0	−5	7
2				

4. The row below the horizontal line and between the bars houses the quotient's coefficients; the remainder will appear in the bottom right corner. Now let's start the algorithm:

(a) copy 4 (the leading coefficient in $A(x)$) in the first place below the horizontal line;

	4	0	-5	7
2		8		
	4			

(b) write, under the coefficient of x^2, the product 2 times 4, then add it to the number above it and write the result below the bar;

	4	0	-5	7
2		8		
	4	8		

(c) repeat this procedure until you obtain the remainder:

	4	0	-5	7
2		8	16	22
	4	8	11	29

Detailed exercises

1. $x^3 - 3x^2 + 2x$
   There is a common factor. Answer: $x(x^2 - 3x + 2) = x(x-1)(x-2)$

2. $x^3 + x^2 + x + 1$
   There are partial common factors. Answer: $x(x^2 + 1) + 1(x^2 + 1) = (x+1)(x^2+1)$

3. $3x^3 - 9x - 6$
   $3x^3 - 9x - 6 = 3(x^3 - 3x - 2)$ is divisible by $x - 2$

x^3		$-3x$	-2		x	-2
x^3	$-2x^2$			x^2	$+2x$	$+1$
	$2x^2$	$-3x$	-2			
	$2x^2$	$-4x$				
		x	-2			
		x	-2			
			0			

Answer: $3(x^3 - 3x - 2) = 3(x^2 + 2x + 1)(x - 2) = 3(x+1)^2(x-2)$

4. $x^3 - 8$
   is divisible by $x - 2$

x^3			-8		x	-2
x^3	$-2x^2$			x^2	$+2x$	$+4$
	$2x^2$		-8			
	$2x^2$	$-4x$				
		$4x$	-8			
		$4x$	-8			
			0			

Answer: $(x^2 + 2x + 4)(x - 2)$

5. Find all numbers α for which $(x - \alpha)$ divides the expression

$$(x^3 - 9x)(x + 5)^2(x^2 - 4)^3.$$

$(x^3 - 9x)(x+5)^2(x^2-4)^3 = x(x-3)(x+3)(x+5)^2(x-2)^3(x+2)^3$ Answer: $0, -3, 3, -5, -2, 2$

5 EXERCISES – POLYNOMIALS AND RATIONAL FUNCTIONS

TRUE OR FALSE?

1. The coefficient of t^2 in the polynomial $3t - 2 - t^3$ is 0. **T** **F**
2. $x^3 - 3x - 2$ has only the roots -1 and 2. **T** **F**
3. $\dfrac{3x-y}{y-x-1} - 1 = \dfrac{3x-y-1}{y-x-1}$. **T** **F**
4. $\dfrac{x-y}{2x} + \dfrac{1-y}{2y} = \dfrac{x-y^2}{2xy}$. **T** **F**
5. $(3x-y)(-3x-y) = -9x^2 - y^2$. **T** **F**
6. The remainder of the division between $3x^5 - 4x + 7$ and $x + 2$ is 95. **T** **F**
7. The constant polynomial 4 divides the polynomial $16x^3 - 12x^2 + 4x + 1$. **T** **F**
8. $x - 1$ divides $x^3 - 4x^2 + 5x + 4$. **T** **F**
9. Dividing $x^{18} - 1$ by $x + 1$ gives no remainder. **T** **F**
10. Dividing $2x - 2$ by $\sqrt{5}$ gives remainder 0. **T** **F**

EXERCISE 1

Compute

1. $\left(\dfrac{1}{x} - \dfrac{1}{y}\right)\left(\dfrac{1}{y} + \dfrac{x-y}{x^2+y^2}\right)\left(\dfrac{y}{x-y} + \dfrac{x}{x+y}\right)$
2. $\left(\dfrac{x-2}{x^2+5x+6} - \dfrac{x-1}{x^2+4x+3}\right) \Big/ \dfrac{2x}{x^2+3x+2}$

EXERCISE 2

Compute remainder and quotient of the following expressions

1. $\dfrac{x^4+2x^3-14x^2+2x-15}{x^2+1}$
2. $\dfrac{2x^5+12x^3+x^2+6}{x^3-4}$
3. $\dfrac{x^3-x^2-52x+160}{x-5}$
4. $\dfrac{(x^2+5)(x^2-9)}{2x^2+3}$

EXERCISE 3

Find the numbers that verify the equality, when substituted to A and B:

$$\frac{3x}{4-x^2} = \frac{A}{2-x} + \frac{B}{2+x}$$

Exercise 4

For which value of the parameter a does $x - a$ divide $(x^2 - 4)(x - 8)^3$?

Exercise 5

Factorize:

1. $x^6 - y^6$
2. $x^4 + 1 - 2x^2$
3. $5a^4 + 5a$
4. $a^4 - 3a^3 - 10a^2$
5. $x^3 + x^2 - 4x - 4$
6. $(x^3 - 8)(x + 2) + (x^3 + 8)(x - 2)$
7. $2x^3 - 5x^2 - 28x + 15$
8. $x^3 - 6x^2 - 2x + 7$
9. $x^4 - 4x^3 - 4x^2 + 28x - 21$
10. $x^2 + 17x^+16$
11. $(x + 1)^2 - (x - 1)^2$

6 SOLUTIONS

True or false?

True: 1, 2, 4, 7, 9, 10

Exercise 1

1. $-\frac{1}{y^2}$ 2. $-\frac{1}{x+3}$

Exercise 2

1. $Q = x^2 + 2x - 15 \quad R = 0$
2. $Q = 2x^2 + 12 \quad R = 9x^2 + 54$
3. $Q = x^2 + 4x - 32 \quad R = 0$
4. $Q = \frac{1}{2}x^2 - \frac{11}{4} \quad R = -\frac{147}{4}$

Exercise 3

$A = 3/2, \quad B = -3/2$

Exercise 4

$a \in \{-2, 2, 8\}$

Exercise 5

1. $(x+y)(x-y)(x^2+xy+y^2)(x^2-xy+y^2)$
2. $(x+1)^2(x-1)^2$
3. $5a(a+1)(a^2-a+1)$
4. $a^2(a+2)(a-5)$
5. $(x+1)(x+2)(x-2)$
6. $2(x+2)(x-2)(x^2+4)$
7. $(x+3)(x-5)(2x-1)$
8. $(x-1)\left(x-\frac{5+\sqrt{53}}{2}\right)\left(x-\frac{5-\sqrt{53}}{2}\right)$
9. $(x-1)(x-3)\left(x-\sqrt{7}\right)\left(x+\sqrt{7}\right)$
10. $(x+1)(x+16)$
11. $4x$