Trigonometric functions

Bridging course in mathematics

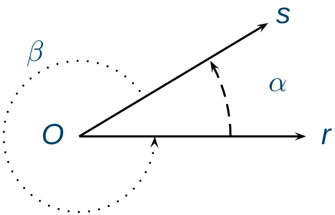
Lesson 5

Outline

Angles

Consider the two half-lines r, s emanating from a point O on the plane

- (r, s) is an ordered pair
- counter-clockwise rotations
- α is a convex angle
- β is a concave angle

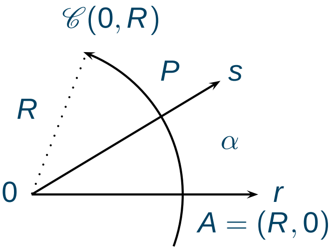


Special configurations

- if $r = s \implies \alpha = 0°$
- if $r \perp s \implies \alpha = 90°$ ie, r and s are **orthogonal**
- if $s = -r \implies \alpha = 180°$

In the Cartesian frame system with origin 0

- r = positive x-semi-axis
- $\mathscr{C}(0, R)$ = circle with centre 0 and radius $R > 0$
- $P = \mathscr{C}(0, R) \cap s$
- AP = length of arc from A to P



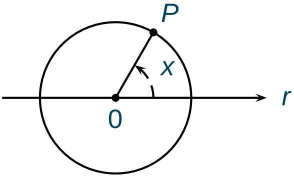
Measure in **radians** of the angle (r, s): $\alpha = AP/R$

Basic angles

degrees	$0°$	$30°$	$45°$	$60°$	$90°$	$180°$	$270°$	$360°$
radians	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π

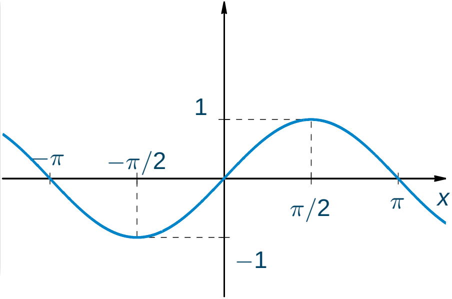
Trigonometric functions : $y = \sin x$

- $\mathscr{C}(0,1)$ unit circle $(R = 1)$
- $P = (\cos x, \sin x)$



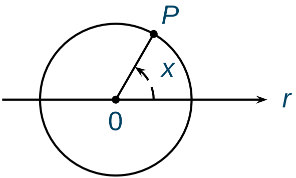
The sine function

- $\mathrm{dom}(f) = \mathbb{R}$, $\mathrm{im}(f) = [-1, 1]$
- periodic $T = 2\pi$
- $\sin(0) = \sin(\pi) = 0$
- $\sin\left(\frac{\pi}{2}\right) = 1$ and $\sin\left(\frac{3\pi}{2}\right) = -1$
- $\sin(-x) = -\sin(x)$ (odd)



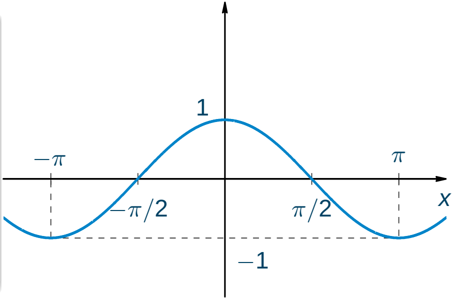
Trigonometric functions : $f(x) = \cos x$

- $\mathscr{C}(0,1)$ unit circle $(R = 1)$
- $P = (\cos x, \sin x)$

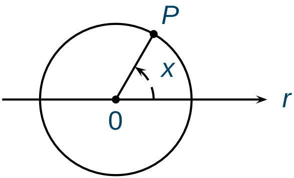


The cosine function

- $\mathrm{dom}(f) = \mathbb{R}$, $\mathrm{im}(f) = [-1, 1]$
- periodic $T = 2\pi$
- $\cos\left(\frac{\pi}{2}\right) = \cos\left(\frac{3\pi}{2}\right) = 0$
- $\cos(0) = 1$ and $\cos(\pi) = -1$
- $\cos(-x) = \cos(x)$ (even)

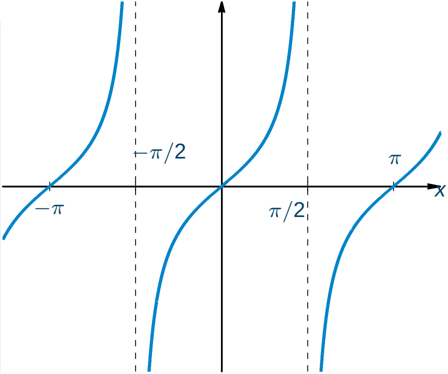


- $\mathscr{C}(0,1)$ unit circle $(R = 1)$
- $P = (\cos x, \sin x)$



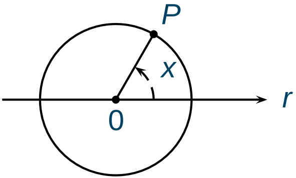
The tangent function

- $y = \tan x = \dfrac{\sin x}{\cos x}$
- $\mathrm{dom}(f) = \mathbb{R} \setminus \{\frac{\pi}{2} + k\pi,\ k \in \mathbb{Z}\}$
- $\mathrm{im}(f) = \mathbb{R}$
- periodic $T = \pi$
- $\tan(0) = 0$
- $\tan(-x) = -\tan(x)$ (odd)



Fundamental identity

- $\mathscr{C}(0,1)$ unit circle $(R = 1)$
- $P = (\cos x, \sin x)$



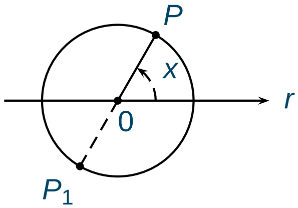
$$\boldsymbol{\sin^2 x + \cos^2 x = 1}$$

Basic angles and values

$\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$	$\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$	$\sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$
$\cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$	$\cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$	$\cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$
$\tan\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{3}$	$\tan\left(\frac{\pi}{3}\right) = \sqrt{3}$	$\tan\left(\frac{\pi}{4}\right) = 1$

Angles and symmetries I

- $\mathscr{C}(0,1)$ unit circle $(R=1)$
- $P = (\cos x, \sin x)$

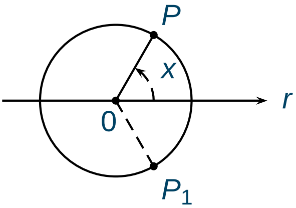


$$P_1 = (\cos(x+\pi), \sin(x+\pi)) = (-\cos x, -\sin x)$$

π–shift

- $\sin(\pi + x) = -\sin(x)$
- $\cos(\pi + x) = -\cos(x)$
- $\tan(\pi + x) = \tan(x)$

- $\mathscr{C}(0,1)$ unit circle $(R = 1)$
- $P = (\cos x, \sin x)$



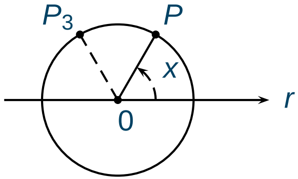
$$P_2 = (\cos(2\pi - x), \sin(2\pi - x)) = (\cos x, -\sin x)$$

π–reflection

- $\sin(2\pi - x) = \sin(-x) = -\sin(x)$
- $\cos(2\pi - x) = \cos(-x) = \cos(x)$
- $\tan(2\pi - x) = \tan(-x) = -\tan(x)$

Angles and symmetries III

- $\mathscr{C}(0,1)$ unit circle $(R = 1)$
- $P = (\cos x, \sin x)$



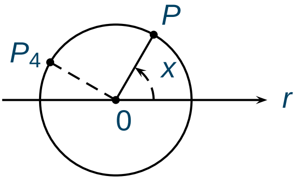
$$P_3 = (\cos(\pi - x), \sin(\pi - x)) = (-\cos x, \sin x)$$

$\frac{\pi}{2}$–reflection

- $\sin(\pi - x) = \sin(x)$
- $\cos(\pi - x) = -\cos(x)$
- $\tan(\pi - x) = -\tan(x)$

Angles and symmetries IV

- $\mathscr{C}(0,1)$ unit circle $(R=1)$
- $P=(\cos x, \sin x)$

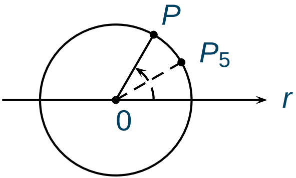


$$P_4=\left(\cos\left(\frac{\pi}{2}+x\right), \sin\left(\frac{\pi}{2}+x\right)\right)=(-\sin x, \cos x)$$

$\frac{\pi}{2}$–shift

- $\sin\left(\frac{\pi}{2}+x\right)=\cos x$
- $\cos\left(\frac{\pi}{2}+x\right)=-\sin x$
- $\tan\left(\frac{\pi}{2}+x\right)=-\cot x$

- $\mathscr{C}(0,1)$ unit circle $(R = 1)$
- $P = (\cos x, \sin x)$



$$P_5 = \left(\cos\left(\frac{\pi}{2} - x\right), \sin\left(\frac{\pi}{2} - x\right)\right) = (\sin x, \cos x)$$

$\frac{\pi}{4}$–reflection

- $\sin\left(\frac{\pi}{2} - x\right) = \cos x$
- $\cos\left(\frac{\pi}{2} - x\right) = \sin x$
- $\tan\left(\frac{\pi}{2} - x\right) = \cot x$

Addition formulas

Addition

$$\sin(x+y) = \sin x \cos y + \sin y \cos x$$
$$\cos(x+y) = \cos x \cos y - \sin x \sin y$$
$$\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$$

Subtraction: write $x - y = x + (-y)$, use symmetries to get

$$\sin(x-y) = \sin x \cos y - \sin y \cos x$$

Duplication: write $2x = x + x$ to get

$$\sin(2x) = 2 \sin x \cos x$$
$$\cos(2x) = \cos^2 x - \sin^2 x$$

Addition formulas revisited

Product-to-sum

$$\begin{aligned}
\sin x \cos y &= \tfrac{1}{2}(\sin(x+y) + \sin(x-y)) \\
\sin x \sin y &= \tfrac{1}{2}(\cos(x-y) - \cos(x+y)) \\
\cos x \cos y &= \tfrac{1}{2}(\cos(x+y) + \cos(x-y))
\end{aligned}$$

Sum-to-product

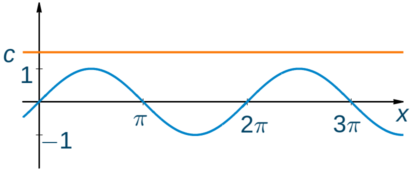
$$\begin{aligned}
\sin x \pm \sin y &= 2 \sin \tfrac{x \pm y}{2} \cos \tfrac{x \mp y}{2} \\
\cos x + \cos y &= 2 \cos \tfrac{x+y}{2} \cos \tfrac{x-y}{2} \\
\cos x - \cos y &= -2 \sin \tfrac{x+y}{2} \sin \tfrac{x-y}{2}
\end{aligned}$$

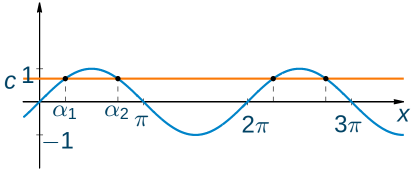
Look for

$$x \in \mathbb{R} : \quad \sin(x) = c$$

- if $|c| > 1 \implies$ no solution
- if $|c| \leq 1 \implies$ infinitely many solutions of the form

$$\begin{cases} \alpha_1 \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right], & \alpha_1 + 2k\pi \\ \alpha_2 = \pi - \alpha_1, & \alpha_2 + 2k\pi \end{cases}$$



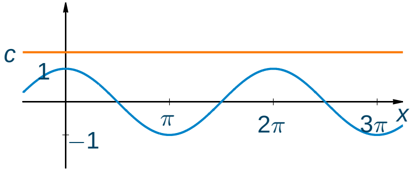


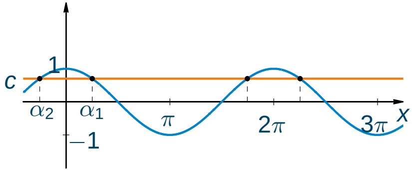
Look for

$$x \in \mathbb{R}: \quad \cos(x) = c$$

- if $|c| > 1 \Longrightarrow$ no solution
- if $|c| \leq 1 \Longrightarrow$ infinitely many solutions of the form

$$\begin{cases} \alpha_1 \in [0, \pi], & \alpha_1 + 2k\pi \\ \alpha_2 = 2\pi - \alpha_1, & \alpha_2 + 2k\pi \end{cases}$$





Linear equations in sin and cos

Given $a, b, c, \in \mathbb{R}$, look for

$$x \in \mathbb{R} : \quad a \sin(x) + b \cos(x) = c$$

Set

$$t = \sin(x), \quad s = \cos(x)$$

and solve in t and s

$$\begin{cases} at + bs & = c \\ t^2 + s^2 & = 1 \end{cases}$$

If $(\bar{t}, \bar{s})$ are solutions, find $x \in \mathbb{R}$:

$$\begin{cases} \sin(x) = \bar{t} \\ \cos(x) = \bar{s} \end{cases}$$

Homogeneous eqn.s of degree 2

Given $a, b, c, \in \mathbb{R}$, look for $x \in \mathbb{R}$:

$$a \sin^2(x) + b \cos^2(x) + c \sin(x) \cos(x) = d$$

$a = d$

- $d = d(\cos^2(x) + \sin^2(x))$
- solve

$$\cos(x) \left[(b - d) \cos x + c \sin x \right] = 0$$

$a \neq d$

- $d = d(\cos^2(x) + \sin^2(x))$
- divide by $\cos^2(x)$
- solve

$$(a - d)t^2 + ct + (b - d) = 0 \rightsquigarrow \bar{t}$$

- find solutions $x \in \mathbb{R}$ to

$$\tan(x) = \bar{t}$$

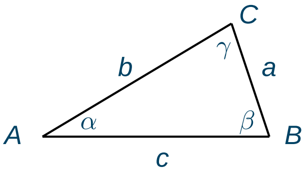
Broad indications

- write all trigonometric functions using **only sine and cosine**
- express everything in terms of the **same angle**
- in absence of constant summands, **factorize** the equation
- examine the **admissible values** carefully, when performing algebraic operations such as dividing by a function
- sketch a **picture**
- interpret geometrically on the unit circle

Properties of triangles

Sine rule

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma}$$



Cosine rule

$$a^2 = b^2 + c^2 - 2bc\cos\alpha$$
$$b^2 = a^2 + c^2 - 2ac\cos\beta$$
$$c^2 = a^2 + b^2 - 2ab\cos\gamma$$

Case: $\alpha = \frac{\pi}{2}$

$$b = a\sin\beta = a\cos\gamma$$
$$c = a\sin\gamma = a\cos\beta$$
$$b = c\tan\beta$$
$$c = b\tan\gamma$$