Exponentials and logarithms

Bridging course in mathematics

Lesson 4

1859-2009

150 anni di Cultura Politecnica

POLITECNICO DI TORINO 1859 1906

Outline

The exponential map

Given a real number $a > 0$, one calls **exponential map in base a** the function

$$x \mapsto a^x$$

Domain and range

The exponential map $x \mapsto a^x$ has

- domain $\mathbb{R}$
- range $(0, +\infty)$

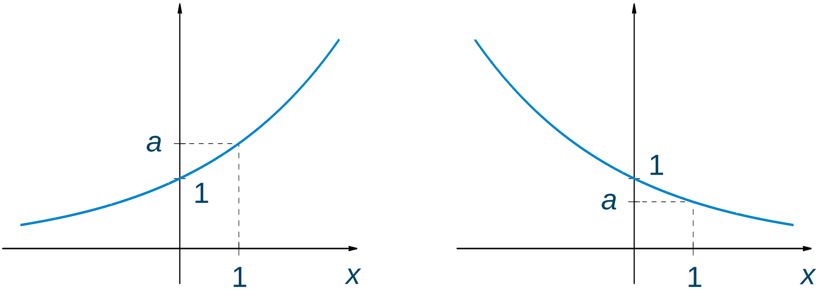
Recall

- for every $a > 0$ and any $x, y \in \mathbb{R}$

$$a^0 = 1 \qquad a^1 = a \qquad a^{-1} = \frac{1}{a} \qquad a^{x+y} = a^x a^y \qquad (a^x)^y = a^{xy}$$

- if $a = 1$, the map is constant: $1^x = 1$

Graph of the exponential map



(a) exponential in base $a > 1$ (b) exponential in base $a < 1$

- The graph passes through the points $(0, 1)$ and $(1, a)$
- If $a > 1$, the function $x \mapsto a^x$ is positive and increasing
- If $a < 1$, the function $x \mapsto a^x$ is positive and decreasing

Bases

- The graph of the exponential map in base a is symmetric to the graph in base $1/a$ with respect to the y-axis

$$y(x) = \left(\frac{1}{a}\right)^x = a^{-x}$$

A special function

- A central role is played by the exponential map in base ***e***

$$\mathbf{y = e^x}$$

As $\boldsymbol{e \approx 2,718}$, the exponential in base ***e*** has an intermediate behaviour between the exponentials in base 2 and 3

Inversion

Given $f(x) = a^x$ with real $a > 0$, and the positive real number y_0, consider the equation $a^x = y_0$

Cases

- $a = 1$
  the equation is solved by any real number if $y_0 = 1$, while it has no solution for $y_0 \neq 1$
- $a \neq 1$
  for any $y_0 > 0$ the equation has one, and only one, solution x_0, called the **logarithm in base** a **of** y_0

The second case defines, for any $y_0 \in (0, \infty)$, a function:

$$y_0 \mapsto \log_a y_0$$

The logarithm function

Given a real number $a > 0$, $a \neq 1$, one calls **logarithm in base a** the map

$$x \mapsto \log_a x$$

Domain and range

The logarithm $y = \log_a x$ has

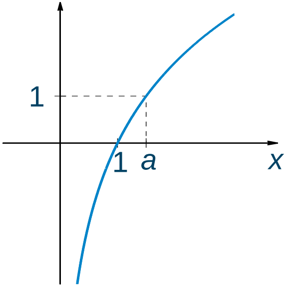
- domain $(0,+\infty)$
- range $\mathbb{R}$

A special function

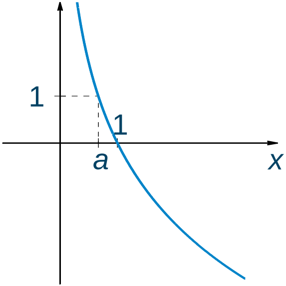
Choosing ***e*** as base defines the so-called natural logarithm, which is customarily denoted by

$$\boldsymbol{y = \ln x}$$

Graph



(c) logarithm in base $a > 1$



(d) logarithm in base $a < 1$

- The graph passes through the points $(1, 0), (a, 1), \left(\frac{1}{a}, -1\right)$
- If $a > 1$ the map is increasing, negative on $(0, 1)$, positive on $(1, \infty)$
- If $a < 1$ the map is decreasing, positive on $(0, 1)$, negative on $(1, \infty)$

Properties

Consider positive reals $a \neq 1$, x, y, and let z be another given real

- $\log_a xy = \log_a x + \log_a y$
- $\log_a \frac{x}{y} = \log_a x - \log_a y$
- $\log_a x^z = z \log_a x$

Furthermore, if b is a positive real $\neq 1$, then the formula of **base change** for logarithms holds:

$$\log_b x = \frac{\log_a x}{\log_a b}$$

The graphs of $y = a^x$ and $y = \log_a x$ (same base) are symmetric to each other with respect to the bisectrix of the first and third quadrant

Hence, if the point (p, q) belongs to the exponential graph, then (q, p) lies on the logarithm graph

Reason

The logarithm and the exponential satisfy the following relationships

- $a^{\log_a y_0} = y_0 \quad \forall y_0 \in (0, +\infty)$
- $\log_a (a^{x_0}) = x_0 \quad \forall x_0 \in \mathbb{R}$

Exponential equations I

Type: $a^{f(x)} = k$ with $a > 0,\ a \neq 1$ and $k \in \mathbb{R}$

Solution: if $k > 0, \quad f(x) = \log_a k$ (what if $k \leq 0$???)

Example

$8 \cdot 2^{x-1} - 2^{x+1} = 16$

$$\begin{aligned} \frac{8 \cdot 2^x}{2} - 2 \cdot 2^x &= 16 \\ (4-2) \cdot 2^x &= 16 \\ 2^x &= 8 \\ x &= 3 \end{aligned}$$

Exponential equations II

Type: $a^{f(x)} = a^{g(x)}$

Solution: $f(x) = g(x)$

Example

$2^{2x^2+x} - 2^{x^3+2x} = 0$

$$\begin{aligned} 2^{2x^2+x} &= 2^{x^3+2x} \\ x(2x+1) &= x(x^2+2) \\ x(-x^2+2x-1) &= 0 \\ x\left(-(x-1)^2\right) &= 0 \\ x=0 \quad \text{or} &\quad x=1 \end{aligned}$$

Exponential equations III

Type: $a^{f(x)} = b^{g(x)},\ b > 0,\ b \neq 1$

Solution: write $b^{g(x)} = a^{g(x) log_a b}$, then apply $\log_a$

Example

$2^{x+1} = 5^{1-x}$

$$\begin{aligned} 2^{x+1} &= 2^{(1-x)\log_2 5} \\ x+1 &= (1-x)\log_2 5 \\ x(1+\log_2 5) &= \log 5 - 1 \\ x &= \frac{\log_2 5 - 1}{1+\log_2 5} \end{aligned}$$

Exponential equations III

Another example

$\frac{2^{x+1}5^{x-1}}{3^x} = 2$

$$
\begin{aligned}
2^{x+1}5^{x-1} &= 2 \cdot 3^x \\
\ln 2^x + \ln 5^{x-1} &= \ln 3^x \\
x \ln 2 + x \ln 5 - x \ln 3 &= \ln 5 \\
x &= \frac{\ln 5}{\ln 2 + \ln 5 - \ln 3}
\end{aligned}
$$

Exponential equations IV

Type: $f(a^x) = 0$

Solution: set $a^x = t$, then solve $f(t) = 0$

Example

$2^{2-x} - 2^{3-x} + 2^x = 0$

$$\begin{aligned}
2^2 2^{-x} - 2^3 2^{-x} + 2^x &= 0 \\
(2^2 - 2^3)2^{-x} &= -2^x \qquad \text{substitution: } t = 2^x \\
(2^2 - 2^3)/t &= -t \\
t^2 &= (2^3 - 2^2) = 8 - 4 = 4 = 2^2 \\
2^{2x} &= 2^2 \\
2x &= 2 \\
x &= 1
\end{aligned}$$

Exponential inequalities

Type: $a^{f(x)} > a^{g(x)}, \quad a > 0, \quad a \neq 1$

Solution: if $a > 1, \quad f(x) > g(x)$

if $a < 1, \quad f(x) < g(x)$

Example

$\left(\left(\frac{1}{7}\right)^{x+1}\right)^x > \frac{1}{49}$

$$\begin{aligned} \left(\frac{1}{7}\right)^{(x+1)x} &> \left(\frac{1}{7}\right)^2 \\ (x+1)x &< 2 \\ x^2 + x - 2 &< 0 \\ -2 &< x < 1 \end{aligned}$$

Exponential inequalities

Type: $f(a^x) > c$

Solution: set $a^x = t$, and solve $f(t) > c$

Example

$4^x - 2 \cdot 2^x - 3 \leq 0$

$$\begin{aligned} 2^{2x} - 2 \cdot 2^x - 3 &\leq 0 && \text{substitution } t = 2^x \\ t^2 - 2t - 3 &\leq 0 \\ -1 &\leq t \leq 3 \\ -1 &\leq 2^x \leq 3 \\ x &\leq \log_2 3 \end{aligned}$$

Logarithmic equalities I

Type: $\log_a f(x) = b$ with $a > 0$, $a \neq 1$ and $b \in \mathbb{R}$
Solution: where $f(x) > 0$, $\quad f(x) = a^b$

Warning

It's always necessary to determine the existence domain,

since log is defined only when its argument is strictly > 0

Example

$2 + \log_2 x = \log_2 7$ $\qquad D = (0; +\infty)$

$$\begin{aligned} \log_2 x &= \log_2 7 - 2 \\ x &= 2^{\log_2 7 - 2} \\ x &= \frac{7}{4} \text{ (valid, because } \in D) \end{aligned}$$

Logarithmic equalities I

Example

$\log_4(x+6) + \log_4 x = 2$ $\qquad D = (0; +\infty)$

$$\begin{aligned} \log_4(x^2+6x) &= 2 \\ x^2+6x-16 &= 0 \\ x &= \begin{cases} -8 & \text{(not valid)} \\ 2 \end{cases} \end{aligned}$$

Logarithmic equalities II

Type: $\log_a f(x) = \log_a g(x)$

Solution: where $f(x) > 0$ and $g(x) > 0$, $\quad f(x) = g(x)$

Example

$\log_2 x + \log_{\frac{1}{2}}(x-1) = 3$ $\qquad D = (1; +\infty)$

$$\begin{aligned}
\log_2 x &= \log_2(x-1) + 3 \\
2^{\log_2 x} &= 2^{\log_2(x-1)+3} \\
x &= (x-1)2^3 \\
x &= 8x - 8 \\
7x &= 8 \\
x &= \frac{8}{7} \ \text{(ok)}
\end{aligned}$$

Logarithmic equalities II

Example

$\log_2(x+1) = \log_4(2x+5)$ $\qquad D = (-1; +\infty)$

$$
\begin{aligned}
\log_2(x+1) &= \frac{\log_2(2x+5)}{\log_2 4} \\
\log_2(x+1) &= \frac{1}{2}\log_2(2x+5) \\
\log_2(x+1) &= \log_2(2x+5)^{\frac{1}{2}} \\
x+1 &= \sqrt{2x+5} \\
x^2+2x+1 &= 2x+5 \\
x^2-4 &= 0 \\
x &= \begin{cases} -2 & \text{(not valid)} \\ 2 \end{cases}
\end{aligned}
$$

Logarithmic equalities III

Type: $f(\log_a x) = 0$

Solution: set $\log_a x = t$, then solve $f(t) = 0$

Example

$\log_2^2 x - 2\log_2 x - 3 = 0$ $\qquad D = (0; +\infty)$

$$\begin{aligned}
\log_2^2 x - 2\log_2 x - 3 &= 0 \qquad \text{substitution } t = \log_2 x \\
t^2 - 2t - 3 &= 0 \\
(t-3)(t+1) &= 0 \\
t &= -1 \text{ or } 3 \\
\log_2 x &= -1 \text{ or } 3 \\
x = 1/2 \quad &\text{or} \quad x = 8
\end{aligned}$$

Logarithmic inequalities I

Type: $\log_a f(x) > \log_a g(x)$

Solution: if $a > 1$, $f(x) > g(x)$;

if $a < 1$, $f(x) < g(x)$

Example

$\log_2 x - log_2 3 < \log_2(x+2)$ $D = (0; +\infty)$

$$\begin{aligned} \log_2 \frac{x}{3} &< \log_2(x+2) \\ \frac{x}{3} &< x+2 \\ x &> -3 \end{aligned}$$

and keeping the domain into account, the solution is $x > 0$

Logarithmic inequalities I

Example

$\log_2(x^2+1) > \log_2(2x+4)$ $\qquad D = (-2; +\infty)$

$$
\begin{aligned}
\log_2(x^2+1) &> \log_2(2x+4) \\
x^2+1 &> 2x+4 \\
x^2-2x-3 &> 0 \\
(x-3)(x+1) &> 0 \\
x < -1 \text{ or } x &> 3
\end{aligned}
$$

and because of the domain, the solution is $-2 < x < -1$ or $x > 3$

Logarithmic inequalities II

Type: $f(\log x) > c$

Solution: set $\log x = t$, and solve $f(t) > c$

Example

$\log_2^3 x - 2\log_2 x > 0$ $\qquad D = (0; +\infty)$

$$\begin{aligned} \log_2^3 x - 2\log_2 x &> 0 \qquad \text{substitution } t = \log_2 x \\ t^3 - 2t &> 0 \\ t(t^2-2) &> 0 \\ t > \sqrt{2} \quad &\text{or} \quad -\sqrt{2} < t < 0 \\ x > 2^{\sqrt{2}} \quad &\text{or} \quad 2^{-\sqrt{2}} < x < 1 \end{aligned}$$