Equations and Inequalities

Bridging course in mathematics

Lesson 3

1859-2009
150 anni di Cultura Politecnica

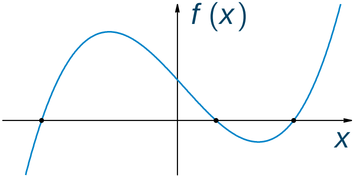
Outline

1. Equations
2. Inequalities
3. Types
   - linear
   - quadratic
   - fractional
   - with absolute values
   - irrational

Understanding equations

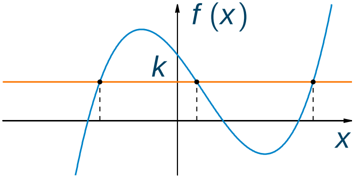
- The **zeroes** of a map f, ie the elements in $\operatorname{dom} f$ at which f takes the value zero, are the **solutions** to the equation

$$f(x) = 0$$



- The elements in the domain at which f assumes a value $k \in \mathbb{R}$ are the solutions to

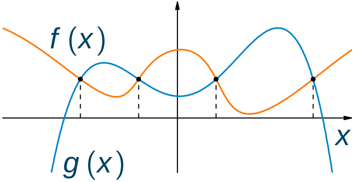
$$f(x) = k$$



- An equation looks like

$$f(x) = g(x)$$

in its most general form



Equivalent equations

Definition

Two equations are said **equivalent** if every solution of the former solves the latter, and conversely, if every solution of the latter also solves the former

Properties

An equation is transformed into an equivalent one by

- adding or subtracting to both sides the same function defined over all of $\mathbb{R}$;
- multiplying or dividing both sides by the same map defined over all of $\mathbb{R}$ and never zero

Remark

By adding (or multiplying by) a function h that is not defined everywhere on $\mathbb{R}$ we might obtain a non-equivalent equation

- Determining where a map f is positive, that is to say, finding the subset of the domain where f assumes positive values, means solving

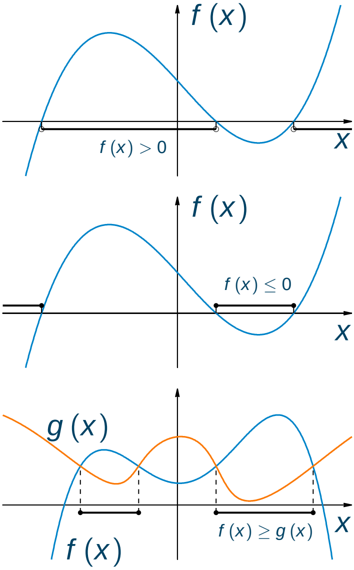
$$f(x) > 0$$

- Similarly, one may find the sets where the map is negative $f(x) < 0$, non-negative $f(x) \geq 0$, or non-positive $f(x) \leq 0$

- General form of an inequality

$$f(x) \geq g(x)$$

(or with $\leq, >, <$)



Equivalent inequalities

Definition

Two inequalities are **equivalent** if every solution of the former solves the latter, and conversely, if every solution of the latter also solves the former

Consider a map $h(x)$ defined on the whole $\mathbb{R}$

- $f(x) < g(x)$ is equivalent to $f(x) \pm h(x) < g(x) \pm h(x)$;
- if $h(x) > 0 \;\; \forall x \in \mathbb{R}$,

  $$f(x) < g(x) \text{ is equivalent to } \begin{cases} f(x)\,h(x) < g(x)\,h(x) & \text{or} \\ f(x)/h(x) < g(x)/h(x) \end{cases}$$

- if $h(x) < 0 \;\; \forall x \in \mathbb{R}$,

  $$f(x) < g(x) \text{ is equivalent to } \begin{cases} f(x)\,h(x) > g(x)\,h(x) & \text{or} \\ f(x)/h(x) > g(x)/h(x) \end{cases}$$

Degree 1

Equations

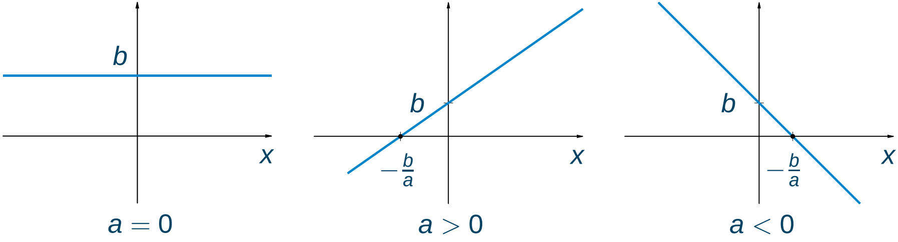
$$ax + b = 0, \quad a, b \in \mathbb{R}$$

		solution
$a = 0$	$b = 0$	$\mathbb{R}$
	$b \neq 0$	$\emptyset$
$a \neq 0$		$x = -\frac{b}{a}$

Inequalities

$$ax + b > 0, \quad a, b \in \mathbb{R}$$

		solution
$a = 0$	$b > 0$	$\mathbb{R}$
	$b \leq 0$	$\emptyset$
$a > 0$		$x > -\frac{b}{a}$
$a < 0$		$x < -\frac{b}{a}$



Degree 2

Consider the quadratic expression

$$ax^2 + bx + c, \quad a, b, c \in \mathbb{R} \quad a \neq 0$$

and define the discriminant Δ

$$\Delta = b^2 - 4ac$$

Equations

$$ax^2 + bx + c = 0$$

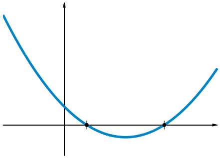
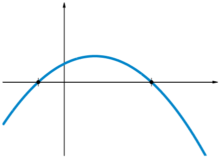
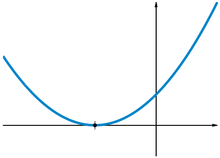
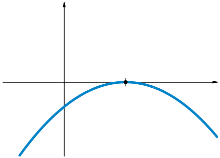
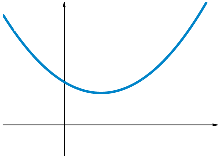
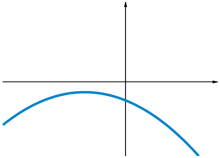
$\Delta > 0$	$x_{1/2} = \frac{-b \pm \sqrt{\Delta}}{2a}$
$\Delta = 0$	$x = \frac{-b}{2a}$
$\Delta < 0$	$\emptyset$

Inequalities

$$ax^2 + bx + c > 0$$

	$a > 0$	$a < 0$
$\Delta > 0$	$\left(-\infty, \frac{-b-\sqrt{\Delta}}{2a}\right) \cup \left(\frac{-b+\sqrt{\Delta}}{2a}, +\infty\right)$	$\left(\frac{-b-\sqrt{\Delta}}{2a}, \frac{-b+\sqrt{\Delta}}{2a}\right)$
$\Delta = 0$	$\mathbb{R} \setminus \left\{-\frac{b}{2a}\right\}$	$\emptyset$
$\Delta < 0$	$\mathbb{R}$	$\emptyset$

Parabolas

	$a > 0$	$a < 0$
$\Delta > 0$		
$\Delta = 0$		
$\Delta < 0$		

Equations

The solutions to

$$\frac{f(x)}{g(x)} = 0$$

are the zeroes of the numerator $f(x)$ which do not make the denominator $g(x)$ vanish

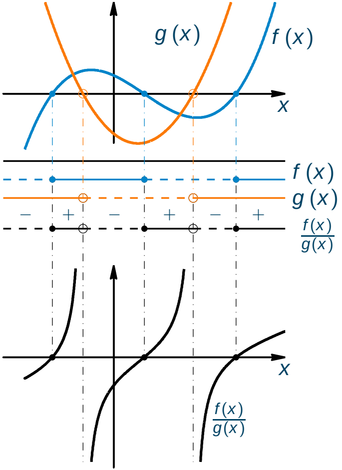
Inequalities

To solve the inequality

$$\frac{f(x)}{g(x)} \geq 0,$$

we study the signs of numerator $f(x)$ and denominator $g(x)$

The overall sign is decided upon by the sign rule



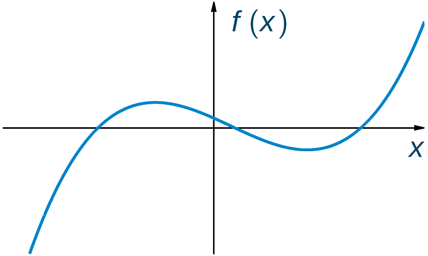
Warning!

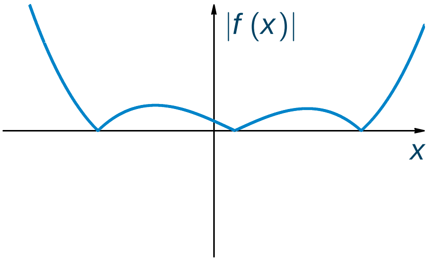
The ratio $\frac{f(x)}{g(x)}$ isn't defined at points where the denominator $g(x)$ vanishes

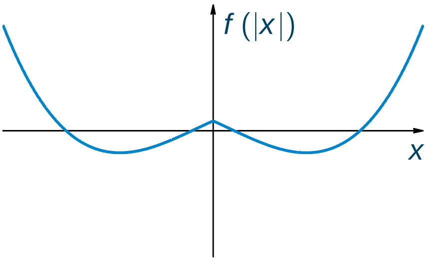
These must be excluded from the set of solutions

Absolute value

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$







Equations with absolute values

$|f(x)| = g(x)$

Solved by the union of the solutions of two systems

$$\left\{ x : \begin{cases} f(x) = g(x) \\ f(x) \geq 0 \end{cases} \right\} \cup \left\{ x : \begin{cases} f(x) = -g(x) \\ f(x) < 0 \end{cases} \right\}$$

Case: $|f(x)| = c$

- $c < 0$: has no solution
- $c = 0$: is equivalent to $f(x) = 0$
- $c > 0$: solutions given by the union of the solutions of two systems

$$\left\{ x : \begin{cases} f(x) = c \\ f(x) \geq 0 \end{cases} \right\} \cup \left\{ x : \begin{cases} f(x) = -c \\ f(x) < 0 \end{cases} \right\}$$

Inequalities with absolute values

$|f(x)| \leq g(x)$

Solved by the union of the solutions of two systems

$$\left\{ x \;:\; \begin{cases} f(x) \leq g(x) \\ f(x) \geq 0 \end{cases} \right\} \cup \left\{ x \;:\; \begin{cases} f(x) \geq -g(x) \\ f(x) < 0 \end{cases} \right\}$$

Case: $|f(x)| \leq c$

- $c < 0$: hasn't got solutions
- $c = 0$: is equivalent to the equation $f(x) = 0$
- $c > 0$: is equivalent to the system $\begin{cases} f(x) \leq c \\ f(x) \geq -c \end{cases}$

Case: $|f(x)| \geq c$

- $c \leq 0$: has solutions given by any $x \in \mathbb{R}$
- $c > 0$: is solved by the union of the solutions of two inequalities

$$\{x \,:\, f(x) \geq c\} \cup \{x \,:\, f(x) \leq -c\}$$

Irrational equations

Equalities

To solve

$$\sqrt[n]{f(x)} = \sqrt[m]{g(x)}$$

- determine existence domain
- raise to the right power = $lcm(n, m)$

Example

$\sqrt[3]{x^3 + 4} - 1 = x$ $D = \mathbb{R}$

$$\begin{aligned} \sqrt[3]{x^3+4} &= x+1 \\ x^3+4 &= (x+1)^3 \end{aligned}$$

$$x_1 = \frac{-1+\sqrt{5}}{2} \quad \text{and} \quad x_2 = \frac{-1-\sqrt{5}}{2}$$

Irrational equations

Examples

$\sqrt{2x-1} = x-2$ $\qquad D = [\frac{1}{2}, +\infty) \cap [2, +\infty)$

$$2x - 1 = (x-2)^2$$

$$x_1 = 1 \quad \text{and} \quad x_2 = 5$$

$$\text{not valid} \qquad \text{valid}$$

$\sqrt{x-1} + \sqrt{x+1} = \sqrt{6-x}$ $\qquad D = [1, 6]$

$2\sqrt{(x-1)(x+1)} = 6 - 3x$ $\qquad \tilde{D} = [1, 2]$

$$4x^2 - 4 = 36 - 36x + 9x^2$$

$$x_1 = \frac{18-2\sqrt{31}}{5} \quad \text{and} \quad x_2 = \frac{18+2\sqrt{31}}{5}$$

$$\text{valid} \qquad \text{not valid}$$

Behaviour of

$$\sqrt[n]{f(x)} < \sqrt[m]{g(x)}$$

depends on parity of roots

For simplicity let's consider $m = 1$

Odd roots

$$\sqrt[2k+1]{f(x)} < g(x) \quad \Longleftrightarrow \quad f(x) < g(x)^{2k+1}$$

Example

$\sqrt[3]{2-x} = x$ $\qquad D = \mathbb{R}$

$$\begin{aligned} 2-x &= x^3 \\ x &= 1 \end{aligned}$$

Irrational inequalities

Even roots

$$\sqrt[2k]{f(x)} < g(x) \iff \begin{cases} f(x) \geq 0 \\ g(x) > 0 \\ f(x) < g(x)^{2k} \end{cases}$$

$$\sqrt[2k]{f(x)} > g(x) \iff \begin{cases} g(x) < 0 \\ f(x) \geq 0 \end{cases} \quad \text{or} \quad \begin{cases} g(x) \geq 0 \\ f(x) > g(x)^{2k} \end{cases}$$

Example

$\sqrt{x-1} > 12 - 2x$

$$\begin{cases} 12 - 2x < 0 \\ x - 1 \geq 0 \end{cases} \quad \text{or} \quad \begin{cases} 12 - 2x \geq 0 \\ x - 1 > (12 - 2x)^2 \end{cases}$$

$$(6, +\infty) \quad \cup \quad (5, 29/4)$$

$$x \in (5, +\infty)$$