Real functions of one real variable

Bridging course in mathematics

Lesson 2

Outline

1. Preliminary definitions
2. Elementary functions
   - polynomial maps
   - rational maps
   - irrational maps
3. Transformations in the plane and graphs

Preliminary definitions

Functions / maps

A **real map of one real variable**

$$f : \mathbb{R} \to \mathbb{R}$$

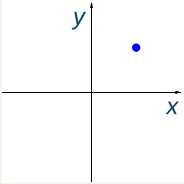
is rule that associates to a number $x \in \mathbb{R}$ **at most** one number $y \in \mathbb{R}$

Points on the plane

This determines points

$$(x_0, y_0) \in \mathbb{R} \times \mathbb{R} = \mathbb{R}^2$$

where $y_0 = f(x_0)$



Preliminary definitions

Graph

The **graph** of a function $f(x)$ is the set of

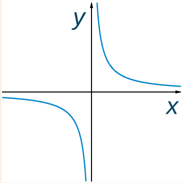
$$(x, y) \in \mathbb{R} \times \mathbb{R} = \mathbb{R}^2 \text{ such that } y = f(x)$$

- $\mathrm{dom}(f) = \{x \in \mathbb{R} : f \text{ is well defined }\}$ is the **domain**
- $\mathrm{im}(f) = \{y \in \mathbb{R} : \exists x \in \mathrm{dom}(f) : y = f(x)\}$ is the **range**

Example: hyperbola

$f(x) = 1/x$

- $\mathrm{dom}\, f = \mathbb{R} \setminus \{0\}$
- $\mathrm{im}\, f = \mathbb{R} \setminus \{0\}$



Polynomial functions

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0$$

$$a_0, \ldots, a_n \in \mathbb{R}$$

- $\mathrm{dom}(f) = \mathbb{R}$

Special cases:

- if $n = 0 \rightsquigarrow$ constant maps
- if $n = 1 \rightsquigarrow$ linear or affine maps
- if $n = 2 \rightsquigarrow$ quadratic maps
- if $a_n = 1$, $a_{n-1} = \ldots = a_0 = 0 \rightsquigarrow$ power functions $f(x) = x^n$

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Constant maps

$$f(x) = q, \quad q \in \mathbb{R}$$

$$\mathrm{dom}(f) = \mathbb{R} \qquad \mathrm{im}(f) = \{q\}$$

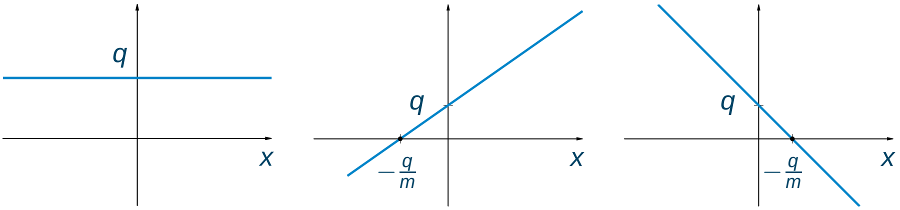
Linear and affine maps

$$f(x) = mx + q, \quad m, q \in \mathbb{R}$$

m = slope $\quad q$ = y-intercept

$$\mathrm{dom}(f) = \mathbb{R}$$

$$\mathrm{im}(f) = \mathbb{R} \quad (m \neq 0)$$



$m = 0 \qquad m > 0 \qquad m < 0$

Quadratic polynomials

$$f(x) = ax^2 + bx + c, \quad a, b, c \in \mathbb{R} \quad a \neq 0$$

$$\Delta = b^2 - 4ac \quad \text{discriminant}$$

$$V = \left(-\frac{b}{2a}; -\frac{\Delta}{4a}\right) \quad \text{vertex}$$

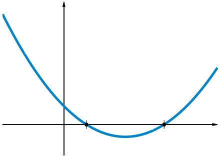
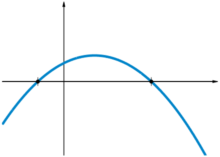
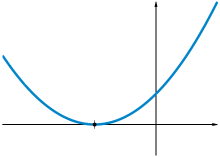
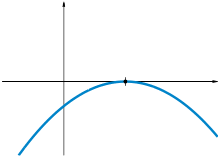
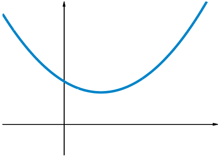
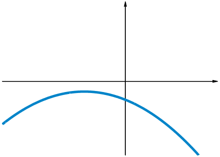
Intersections with x-axis

$$f(x) = 0$$

$\Delta > 0$	$x_{1/2} = \frac{-b \pm \sqrt{\Delta}}{2a}$
$\Delta = 0$	$x = -\frac{b}{2a}$
$\Delta < 0$	$\emptyset$

Convexity

$a > 0$	convex	$\mathrm{im}(f) = \left[-\frac{\Delta}{4a}, +\infty\right)$
$a < 0$	concave	$\mathrm{im}(f) = \left(-\infty, -\frac{\Delta}{4a}\right]$

	$a > 0$	$a < 0$
$\Delta > 0$		
$\Delta = 0$		
$\Delta < 0$		

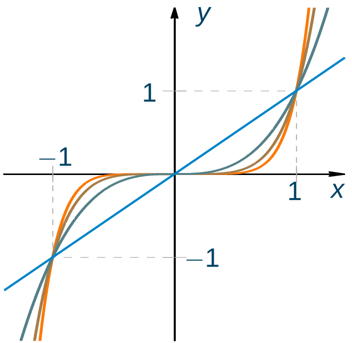
Powers

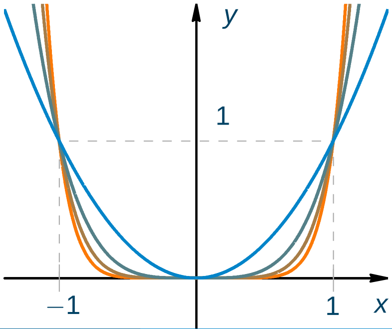
$f(x) = x^n, \quad n \in \mathbb{N}$

$P = (0,0) \quad Q = (1,1)$

Properties

	dom(f)	im(f)	symmetry
n odd	$\mathbb{R}$	$\mathbb{R}$	$f(-x) = -f(x)$
n even	$\mathbb{R}$	$\mathbb{R}^+ \cup \{0\}$	$f(-x) = f(x)$





Rational maps

Given polynomials $p(x)$ and $q(x)$,

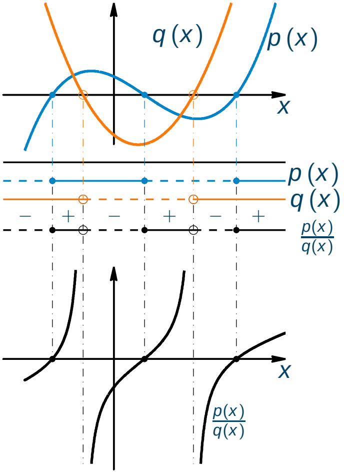
$$f(x) = \frac{p(x)}{q(x)}$$

has $\text{dom}(f) = \mathbb{R} \setminus \{x : q(x) = 0\}$

Intersections with the x-axis

Determined by the points where the numerator vanishes

$$x \in \mathbb{R} : p(x) = 0$$



Positivity

$f(x) > 0$ on the intervals where numerator and denominator have the same sign

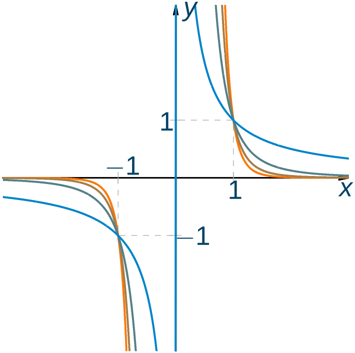
Inverse powers

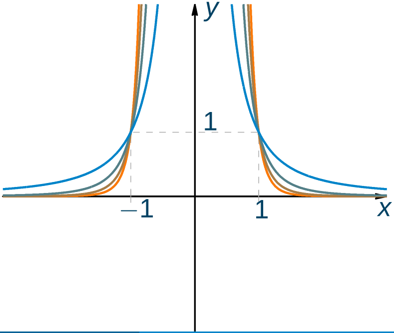
$$f(x) = \frac{1}{x^n}, \quad n \in \mathbb{N}$$

$$\text{dom}(f) = \mathbb{R} \setminus \{0\} \quad Q = (1, 1)$$

Properties

	im(f)	symmetry
n odd	$\mathbb{R} \setminus \{0\}$	$f(-x) = -f(x)$
n even	$\mathbb{R}^+$	$f(-x) = f(x)$





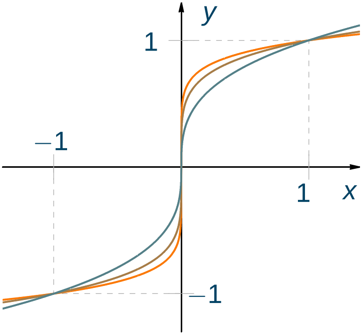
Elementary functions

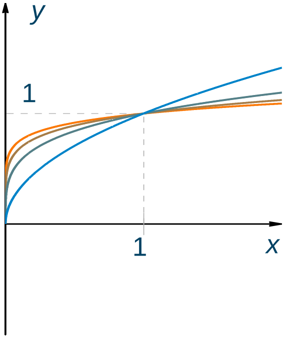
Root functions

$$f(x) = \sqrt[n]{x}, \quad n \in \mathbb{N}$$

Properties

	dom(f)	im(f)	symmetry
n odd	$\mathbb{R}$	$\mathbb{R}$	$f(-x) = -f(x)$
n even	$\mathbb{R}^+ \cup \{0\}$	$\mathbb{R}^+ \cup \{0\}$	none





Moving graphs

Horizontal translation

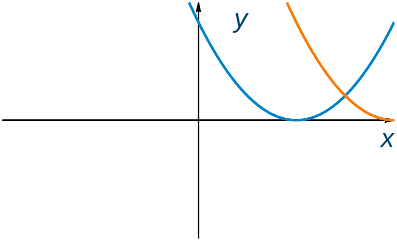
$$y = f(x) \rightsquigarrow y = f(x - p)$$

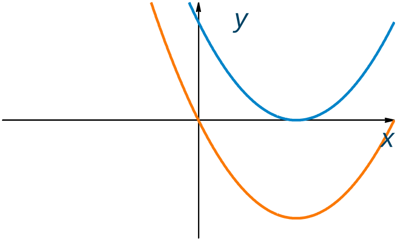
	direction
$p > 0$	to the right
$p < 0$	to the left

Vertical translation

$$y = f(x) \rightsquigarrow y = f(x) + q$$

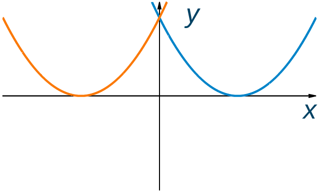
	direction
$q > 0$	upwards
$q < 0$	downwards





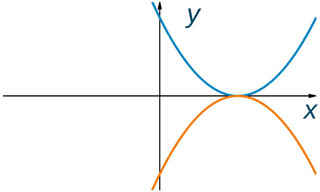
with respect to the y-axis

$$y = f(x) \rightsquigarrow y = f(-x)$$



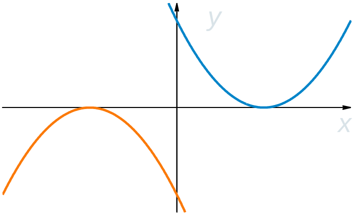
with respect to the x-axis

$$y = f(x) \rightsquigarrow y = -f(x)$$



about the origin

$$y = f(x) \rightsquigarrow y = -f(-x)$$



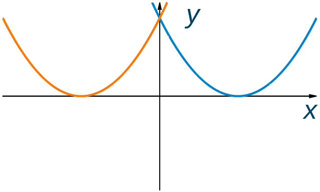
f is called **even** if $f(-x) = f(x)$

f is called **odd** if $f(-x) = -f(x)$

Symmetries

with respect to the y-axis

$$y = f(x) \rightsquigarrow y = f(-x)$$



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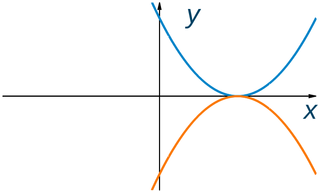

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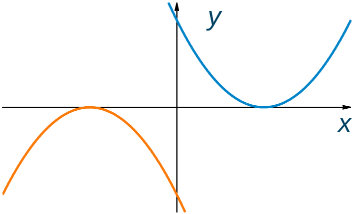
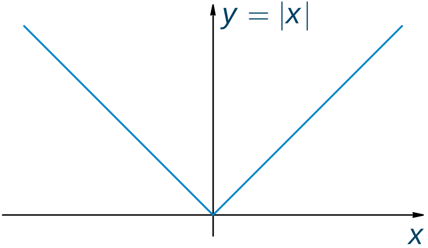


Absolute value function



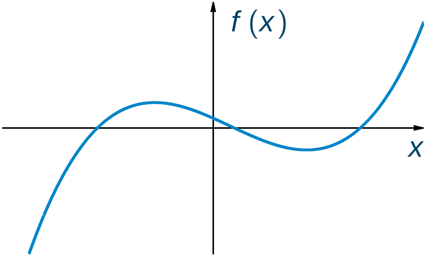
Absolute value

$$y = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

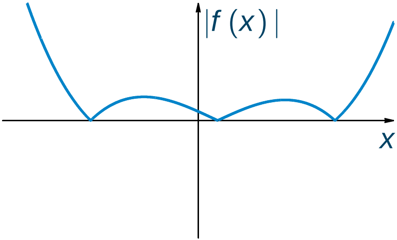
Properties

dom(f)	im(f)	symmetry
$\mathbb{R}$	$\mathbb{R}^+ \cup \{0\}$	$f(-x) = f(x)$

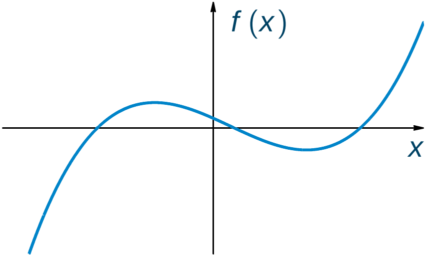
Graph of $|f(x)|$



$$|f(x)| = \begin{cases} f(x) & \text{if } f(x) \geq 0 \\ -f(x) & \text{if } f(x) < 0 \end{cases}$$



Graph of $f(|x|)$



$$f(|x|) = \begin{cases} f(x) & \text{if } x \geq 0 \\ f(-x) & \text{if } x < 0 \end{cases}$$