Polynomials

Bridging course in mathematics

Lesson 1

Outline

Sets of numbers

$\mathbb{R}$, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$

We shall typically work with **real** numbers, whose set is denoted by $\mathbb{R}$

The set of real numbers contains important subsets:

- **natural** numbers, denoted by $\mathbb{N} = \{0, 1, 2, 3, ...\}$
- **integer** numbers, denoted by $\mathbb{Z} = \{..., -3, -2, -1, 0, 1, 2, 3, ...\}$
- **rational** numbers, denoted by $\mathbb{Q}$:
  a rational number can be written as ratio m/n of two integers, with $n \neq 0$

What's a polynomial ?

Definition

- A **real polynomial in** the variable **x** is an algebraic expression of the form
$$A_n(x) = a_0 + a_1 x + \ldots + a_n x^n,$$
where $a_0, a_1, \ldots, a_n$ are real numbers (called the **coefficients** of the polynomial) and $a_n \neq 0$
- The summands are said **monomials**
- The **degree** of a polynomial is the maximum power among the non-zero monomials appearing in the expression

Property

Two polynomials are equal if they have the same degree and the coefficients of the monomials with equal degree coincide

Sum

Adding polynomials

The **sum** of two polynomials is the polynomial obtained by adding the coefficients of the monomials of same degree

$$A_n(x) + B_m(x) = (a_0 + b_0) + (a_1 + b_1)x + ... + (a_m + b_m)x^m$$

Example

$$(x^2 + 2x - 5) + (x^3 - x + 2) =$$
$$(0x^3 + x^2 + 2x - 5) + (x^3 + 0x^2 - x + 2) =$$
$$(0 + 1)x^3 + (1 + 0)x^2 + (2 - 1)x + (-5 + 2) =$$
$$x^3 + x^2 + x - 3$$

Product

Multiplying polynomials

The **product** is the polynomial of the form

$$A_n(x) \cdot B_m(x) = C_{n+m}(x) = c_0 + c_1 x + \dots + c_{n+m} x^{n+m},$$

whose coefficients are given by:

$$c_0 = a_0 b_0, \quad c_1 = a_1 b_0 + a_0 b_1,$$
$$c_k = a_0 b_k + a_1 b_{k-1} + a_2 b_{k-2} + \cdots + a_{k-1} b_1 + a_k b_0$$

In practice

$$(x-1)(x^2+x+1) =$$
$$x^2(x-1) + x(x-1) + (x-1) =$$
$$x^3 - x^2 + x^2 - x + x - 1 =$$
$$x^3 - 1$$

Quotient I

Dividing polynomials

Given polynomials $A_n(x), B_m(x)$ of degrees $n \geq m$, there exist polynomials $Q(x)$ and $R(x)$, called **quotient** and **remainder**, such that:

- the degree of $R(x)$ is less than m;
- $A_n(x) = B_m(x) Q(x) + R(x)$

Definition

If $R(x) = 0$, one says that $B_m(x)$ divides $A_n(x)$

Therefore

The ratio of $A_n(x)$ and $B_m(x)$ can always be written as

$$\boxed{\frac{A_n(x)}{B_m(x)} = Q(x) + \frac{R(x)}{B_m(x)}}$$

where $\deg R < \deg B_m$

Quotient II

The method for dividing two polynomials consists in

dividing monomials by decreasing degree

Example

Let's divide

$$A(x) = 2x^4 + x^3 - x + 2$$

by

$$B(x) = x^2 + 3$$

Example

$2x^4$	$+x^3$	$+0x^2$	$-x$	$+2$	x^2	$+3$
					$Q(x)$	
			$R(x)$			

Example

$2x^4$	$+x^3$	$+0x^2$	$-x$	$+2$	x^2	$+3$
$2x^4$		$+6x^2$			$2x^2$	
	$+x^3$	$-6x^2$	$-x$	$+2$	$\uparrow$	
					$Q(x)$	
					$\leftarrow$ $R(x)$	

Example

$2x^4$	$+x^3$	$+0x^2$	$-x$	$+2$	x^2	$+3$
$2x^4$		$+6x^2$			$2x^2$	$+x$
	$+x^3$	$-6x^2$	$-x$	$+2$	$\uparrow$	
	$+x^3$		$+3x$		$Q(x)$	
		$-6x^2$	$-4x$	$+2$		
					$\leftarrow$	$R(x)$

Example

$2x^4$	$+x^3$	$+0x^2$	$-x$	$+2$	$x^2 \qquad +3$
$2x^4$		$+6x^2$			$2x^2 \quad +x \quad -6$
	$+x^3$	$-6x^2$	$-x$	$+2$	$\uparrow$
	$+x^3$		$+3x$		$Q(x)$
		$-6x^2$	$-4x$	$+2$	
		$-6x^2$		-18	
			$-4x$	$+20$	$\leftarrow \quad R(x)$

$$\frac{A(x)}{B(x)} = Q(x) + \frac{R(x)}{B(x)} = 2x^2 + x - 6 + \frac{-4x + 20}{x^2 + 3}$$

Factorization

Factorizing a polynomial means transforming it into a **product**

Example

$$x^2 - 5x + 6 = (x - 3)(x - 2)$$

A polynomial that cannot be factorized is called **irreducible**;

Real polynomials of degree 1 are irreducible, and also those of degree 2 with negative discriminant.

Useful methods

Finding an overall common factor

$$x^4 - 3x^3 + 5x^2 = x^2\left(x^2 - 3x + 5\right)$$

Finding a partial factor

$$x^4 + a^2x^2 + b^2x^2 + a^2b^2 = x^2(x^2 + a^2) + b^2(x^2 + a^2) =$$

$$(x^2 + a^2)(x^2 + b^2)$$

Useful properties

Factor / remainder theorem

$(x - c)$ divides the polynomial $A_n(x)$ if and only if $A_n(c) = 0$

Consequences

- The binomial $x^n - a^n$ is always divisible by $x - a$;
  if n is even, $x^n - a^n$ is also divisible by $x + a$
- The binomial $x^n + a^n$ is divisible by $x + a$ if n is odd;
  if n is even, $x^n + a^n$ is neither divisible by $x + a$ nor by $x - a$

Remark

For polynomials with **integer coefficients**:

- the possible integer roots should be searched for among the factors with sign of the constant term, including 1;
- the possible rational roots should be looked for among the rationals of the form $\pm p/q$, where p divides the constant term, and q divides the leading coefficient